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New Analytic Solution for Abel Differential Equation

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Abstract: In this paper, a new analytic solution is presented using HPM for a class of nonlinear ODE. The emphasis is on the Abel differential equation. The procedures introduced in this paper are in recursive forms which can be used to obtain the closed form of the solutions, if they are required. The scheme is shown to be highly accurate and in some cases, yields exact solutions in few iterations.

Key words: New analytic solution · Abel differential equation

INTRODUCTION

Large varieties of physical, chemical and biological phenomena are governed by nonlinear evolution equations. Except a limited number of these problems, most of them do not have precise analytical solutions so that they have to be solved using other methods. He's method is a powerful analytical tool for nonlinear problems. This technique provides us with a simple way to ensure the convergence of the solution series, so that we can always get accurate enough approximations. He's iteration method [1-7] is a powerful device for solving various nonlinear equations. Recently this method has attracted a wide class of audiences in all fields of science and engineering [8-19].

In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. This method continuously deforms a simple problem, easy to solve, into the difficult problems under study.

Basic Idea: For the purpose of applications illustration of the methodology of the proposed method, using homotopy perturbation method, we consider the following nonlinear differential equation,

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, $f(r)$ is a known analytic function, B is a boundary condition and Γ is the boundary of the domainΩ

The operator A can be generally divided into two operators, L and N where L is a linear, while N is a nonlinear operator. Equation (1) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

Using the homotopy technique, we construct a homotopy $U(r,p): \Omega \times [0,1] \to \mathbb{R}$ which satisfies:

$$\begin{aligned} H(U,p) &= (1-p)[L(U) - L(u_0))] \\ &+ p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \end{aligned} \tag{4}$$

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(r)] = 0. \tag{5}$$

Where $p \in [0,1]$, is called homotopy parameter and u_0 is an initial approximation for the solution of Eq.(1), which satisfies the boundary conditions. Obviously from Esq. (4) and (5) we will have.

$$H(U,0) = L(U) - L(u_0) = 0, \tag{6}$$

$$H(U,1) = A(U) - f(r) = 0, \tag{7}$$

We can assume that the solution of (4) or (5) can be expressed as a series in p, as follows:

$$U = U_0 + pU_1 + p^2U_2 + \ldots \tag{8}$$

Setting $p = 1$ results in the approximate solution of Eq. (1)

$$u = \lim_{p \to 1} U = U_0 + pU_1 + p^2U_2 + \ldots \tag{9}$$

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Numerical Examples: To illustrate the ability and reliability of the method for Abel differential equation, some examples are provided. The results reveal that the method is very effective and simple.

Example 1: Consider the following Abel differential equation

$$\frac{dy}{dt} = 1 + y^2(t), \quad (10)$$

subject to the initial condition

$$y(0) = 0 \quad (11)$$

With the exact solution $y(t) = \tan t$.

To solve Eq. (10) by homotopy perturbation method, we construct the following homotopy

$$(1-p)\left(\frac{dY}{dt} - \frac{dy_0}{dt}\right) = p\left(\frac{dY}{dt} - 1 - Y^2(t)\right),$$

Suppose the solution of Eq. (10) has the following form

$$Y = Y_0 + pY_1 + p^2Y_2 + \ldots \quad (13)$$

Substituting (13) into (12) and equating the coefficients of the terms with the identical powers of p leads to

$$p^0 : \frac{\partial Y_0}{\partial t} - \frac{\partial y_0}{\partial t} = 0,$$
$$p^1 : \frac{dY_1}{dt} - \frac{dY_0}{dt} - 1 - Y_0{}^2(t) = 0,$$
$$p^2 : \frac{dY_2}{dt} - 2Y_0(t)Y_1(t) = 0,$$
$$\vdots$$
$$p^j : \frac{dY_2}{dt} - \sum_{k=0}^{j-1} Y_k(t)Y_{j-k-1}(t) = 0,$$
$$\vdots$$

We take

$$Y_0 = y_0 = 0. \quad (14)$$

We have the following recurrent equations for $j = 1,2,3,\ldots$.

$$Y_j = \int_0^t \left(\sum_{k=0}^{j-1} Y_k(t)Y_{j-k-1}(t)\right)dt \quad (15)$$

With the aid of the initial approximation given by Eq. (14) and the iteration formula (15) we get the other of component as follows

$$Y_1 = t,$$
$$Y_2 = 0,$$
$$Y_3 = \frac{1}{3}t^3$$
$$\vdots$$

Therefore we have

$$y(t) = t + \frac{1}{3}t^3 + \ldots$$

Example 2: Consider the Abel equation

$$\frac{dy}{dt} = -y(t) + y^2(t), \quad (16)$$

subject to the initial condition

$$y(0) = \frac{1}{2}. \quad (17)$$

With the exact solution $y(t) = \frac{e^{-t}}{1+e^{-t}}$.

To solve Eq. (16) by homotopy perturbation method, we construct the following homotopy

$$(1-p)\left(\frac{dY}{dt} - \frac{dy_0}{dt}\right) = p\left(\frac{dY}{dt} + Y(t) - Y^2(t)\right), \quad (18)$$

Suppose the solution of Eq. (16) has the following form

$$Y = Y_0 + pY_1 + p^2Y_2 + \ldots \quad (19)$$

Substituting (19) into (18) and equating the coefficients of the terms with the identical powers of p leads to

$$p^0 : \frac{\partial Y_0}{\partial t} - \frac{\partial y_0}{\partial t} = 0,$$
$$p^1 : \frac{dY_1}{dt} - \frac{dY_0}{dt} + Y_0(t) - Y_0{}^2(t) = 0,$$
$$p^2 : \frac{dY_2}{dt} + Y_1(t) - 2Y_0(t)Y_1(t) = 0,$$
$$\vdots$$
$$p^j : \frac{dY_j}{dt} + Y_{j-1}(t) - \sum_{k=0}^{j-1} Y_k(t)Y_{j-k-1}(t) = 0,$$
$$\vdots$$

We take

$$Y_0 = y_0 = \frac{1}{2}. \tag{20}$$

We have the following recurrent equations for j = 1,2,3,....

$$Y_j = \int_0^t -Y_{j-1} + \sum_{k=0}^{j-1} Y_k(t) Y_{j-k-1}(t) dt \tag{21}$$

With the aid of the initial approximation given by Eq. (20) and the iteration formula (21) we get the other of component as follows.

$$Y_1 = -\frac{1}{4}t,$$
$$Y_2 = 0,$$
$$Y_3 = \frac{1}{48}t^3$$
$$\vdots$$

Therefore we have

$$y(t) = t + \frac{1}{48}t^3 + \ldots$$

CONCLUSIONS

In this work, we present the analytical approximation to a solution for Abel equations in three different cases. We have achieved this goal by applying He's iteration method. Using the iteration method, it is possible to find the exact solution or a good approximate solution of the equation. It can be concluded that He's iteration method is very powerful and efficient technique for finding exact solutions for wide classes of problems.

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