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Multiple Solutions of a Nonlocal Sturm-Liouville Problem via the Variational Iteration Method

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Abstract: This paper presents a new application of the variational iteration method (VIM) for finding multiple solutions of boundary value problems of ordinary differential equations. In details, we consider numerical solution of the Sturm – Liouville problem with one classical and another nonlocal boundary condition. All possible real eigenvalues can be obtained by starting the VIM algorithm with one initial approximation and Lagrange multiplier. Then, the N th order approximate solution obtained by the VIM is a function of eigenvalues, say λ. Imposing the nonlocal boundary condition, eigenvalues become the roots of a function. By plotting this function, positions of the roots can be seen in the real line. Therefore, the existence, uniqueness and multiplicity of eigenvalues depend on the nonlocal conditions. So, the proposed method is tested via different parameters of the nonlocal conditions. The results are in complete agree with the theory confirming the accuracy of the method.

Key words: Variational iteration method · Nonlocal boundary condition · Sturm-liouville problem

INTRODUCTION

The variational iteration method, was proposed originally by He [1-6]. The method has been successfully applied for numerical solutions of many linear and nonlinear engineering problems [7-18]. The successful application of the method is due to its flexibility, convenience and accuracy. For more useful applications of the method the reader is referred to the references [19-29].

Problems with non-local boundary conditions arise in the modeling of various processes in science and engineering. Theoretical investigations of these kinds of problems can be seen in the literature in many papers such as [30-32]. There exist also many papers on numerical solutions of these kinds of problems, see for example [33-34]. Sturm-Liouville problems with nonlocal conditions are closely linked with boundary value problems for ODEs and PDEs with nonlocal conditions. In this paper, we investigate the following nonlocal Sturm-Liouville problem:

$$-u'' = \lambda u \quad x \in (0,1) \tag{1.1}$$

With the classical condition

$$u(0) = (0), \tag{1.2}$$

and the another nonlocal two-point Samarskii-Bitsadze-type boundary condition

$$u(1) = \gamma u(\xi) \tag{1.3}$$

with $\gamma \in R$ and $\xi \in [0, 1]$. In this paper, we are interested in numerical investigation of this problem by the variational iteration method. It has been proved in [32] that for real γ, a unique negative eigenvalue exists for $\gamma > \frac{1}{\xi}$ and multiple positive eigenvalues exist. This fact has been shown numerically in this paper.

Basics of the VIM: The idea of the method is based on constructing a correction functional by a general Lagrange multiplier and the multiplier is chosen in such a way that its correction solution is improved with respect to the initial approximation or to the trial function.

To illustrate the basic concept of the variational iteration method, we consider the following general nonlinear system.

$$L[u(x)]+N[u(x)] = g(x) \tag{2.4}$$

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Where L is a linear operator, N is a nonlinear operator and $g(x)$ is a given continuous function. According to the variational iteration method, we can construct a correction functional in the form.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[Lu_n(s) + N\tilde{u}_n(s) - g(s)\right]ds$$

Where $u_0(x)$ is an initial approximation with possible unknowns, λ is a Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the n th approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

The successive approximations $u_n(x); n\geq 1$, of the solution $u(x)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function $u_0(x)$. Consequently, the exact solution may be obtained by using.

$$u(x) = \lim_{n\to\infty} u_n(x).$$

For the convergence of the sequence obtained via the VIM and its rate, we recall Banach's theorem:

Theorem 1: (Banach's fixed point theorem)
Assume that X is a Banach space,

$$A : X \to X$$

is a nonlinear mapping and suppose that

$$\left\|A[u] - A[\bar{u}]\right\| \leq \gamma\left\|u - \bar{u}\right\|, \ \forall u, \bar{u} \in X \tag{2.5}$$

for some constant $\gamma < 1$. Then A has a unique fixed point. Furthermore, the sequence

$$u_{n+1} = A[u_n] \tag{2.6}$$

with an arbitrary choice of $u_0 \in X$ converges to the fixed point of A and

$$\left\|u_k - u_1\right\| \leq \left\|u_1 - u_0\right\| \sum_{j=l-1}^{k-2} \gamma^j.$$

According to the above theorem, for the nonlinear mapping

$$A[u] = u(x) + \int_0^x \lambda(s)\left[L u_n(s) + N\tilde{u}_n(s) - g(s)\right]ds,$$

a sufficient condition for the convergence of the variational iteration method is strictly contraction of A. Furthermore, sequence (2.6) converges to the fixed point of A, which is also the solution of the nonlinear differential equation (2.4). In the above theorem, the rate of convergence depends on γ and therefore, in the variational iteration method, the rate of convergence depends on λ.

Simulations for Nonlocal Sturm-Liouville Problem: For the boundary value problem (1.1), according to the variational iteration method, the non-linear terms have to be considered as a restricted variation. So we derive a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left[-u''_n(s) - \lambda\tilde{u}_n(s)\right]ds,$$

and writing the stationary condition of the above correction functional the Lagrange multiplier, can be identified as follows:

$$\lambda = s - x$$

As a result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x) + \int_0^x (s-x)\left[u''_n(s) - \lambda u_n(s)\right]ds. \tag{3.7}$$

Now, we begin with the initial approximation:

$$u_0(x) = x.$$

By the variational iteration formula (3.7), we have

$$u_1(x) = x - \frac{\lambda x^3}{6},$$
$$u_2(x) = x - \frac{\lambda x^3}{6} + \frac{\lambda^2 x^5}{120},$$
$$u_3(x) = x - \frac{\lambda x^3}{6} + \frac{\lambda^2 x^5}{120} + \frac{\lambda^3 x^7}{5040},$$

and so on.

To the N th order approximate solution u_N, which still depends on the eigenvalues λ Condition (1.3) reads.

$$u(1) \approx u_N(1;\lambda) = \gamma u_N(\xi) \tag{3.8}$$

The eigenvalues λ should satisfy the equation (3.8) and so to see the behavior of the eigenvalues and their positions, we can plot the function.

$$f(\lambda) = u_N(1;\lambda) - \gamma u_N(\xi).$$

and any zeros of this function is an eigenvalue of the problem. Now, we study the behavior of the eigenvalues with various parameters γ and ξ.

3.1 Case 1: $\gamma = 0$

In this case, the problem becomes the classical two point sturm-liouville problem. the exact eigenvalues and eigenfunctions are as follows:

$$\lambda_k = (k\pi)^2,\ u_k = \sin(k\pi x),\ k \in N,$$

and therefore all eigenvalues are positive and multiple eigenvalues exist. To see the solution of the VIM, the function f becomes:

$$f(\lambda) = u_N(1;\lambda)$$

The zeros of the function f, which are the eigenvalues of the problem can be seen in the Figure 1. To obtain the numerical approximation of the eigenvalues, one should find the roots of the function f. It can be done easily by an initial approximation from the figure and using suitable software such as MATHEMATICA.

The eigenvalues are obtained as follows:

λ_1 = 9.8696044010 8936,
λ_2 = 39.4784176043 57425
λ_3 = 88.8264396097 9966,
λ_4 = 157.9136704173 8633
λ_5 = 246.7401100289 5259,

and so on.

3.2 Case 2: $\xi = \frac{1}{2}$

After implementing the VIM, the function f is plotted for different values of γ in Figures 2 and 3. In Figure 2, the plots have been done near origin. Because $x \to -\infty$ then $f(x) \to \infty$, one can see from Figure that for $\gamma > \frac{1}{\xi} = 2$ a unique negative eigenvalue exists and for $\gamma = \frac{1}{\xi} = 2, \lambda = 0$ is an eigenvalue. Also for $\gamma < \frac{1}{\xi} = 2$, there is no negative eigenvalue. In all figures, g in the legend means γ.

From Figure 3, it is observed that between two eigenvalues of the case $\gamma > \frac{1}{\xi} = 2$, there exists one eigenvalue for the case $\gamma < \frac{1}{\xi} = 2$.

The first 6 eigenvalues for different γ are presented in Table 1.

Table 1: The eigenvalues with different γ for Case 2

	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$
λ_1	4.386490844928605	0	-3.7050371292351088
λ_2	39.47841760435735	39.47841760435747	39.4784176043574
λ_3	109.66227112319947	157.90316689026628	157.91367041755765
λ_4	157.91367041786359	355.3057584101483	355.305758412833
λ_5	214.938051399216	632.3768793585701	631.6548226231149
λ_6	355.305758839007247	986.9360070287219	986.9396926487374

Table 2: The eigenvalues with different γ for Case 3

	$\gamma = 2$	$\gamma = 4$	$\gamma = 6$
λ^1	3.7645578196677656	0	-2.837513659548815
λ^2	157.91367041744587	157.91367041753804	157.91367041750567
λ^3	537.8918061942263	632.7928884131129	631.6548449966419
λ^4	631.6445415029861	1416.0612747611476	1413.9317209847125

Table 3: The eigenvalues with different γ for Case 4.

	$\gamma = 0.5$	$\gamma = 1.5$	$\gamma = 2.5$
λ^1	6.903116129815923	0	-7.168417703187119
λ^2	34.99371560874429	29.92560160385297	27.47118668127791
λ^3	88.82643960980309	88.82643960980923	88.82643960980508
λ^4	167.28854458521778	179.00101722631928	185.18473050672966
λ^5	263.1590504996487	355.32016939148554	355.3057584188885
λ^6	355.3057584125378	591.4616873957532	580.3691417617187

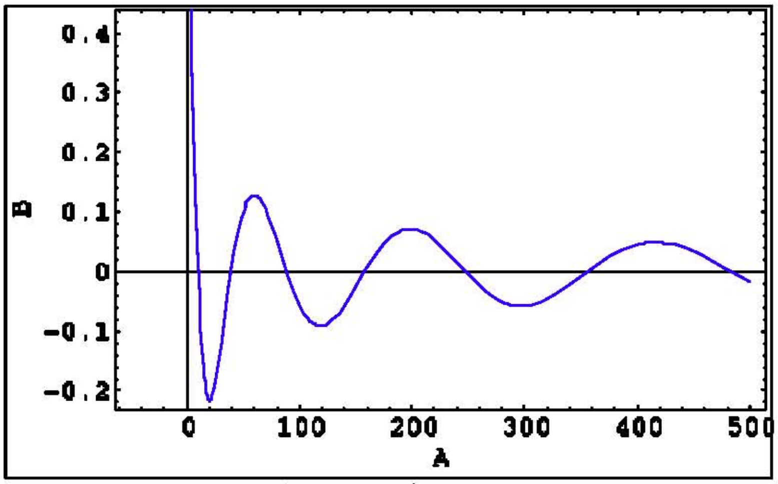

Fig. 1: Plot of the function f for Case 1, B= $f(\lambda)$ and A= λ.

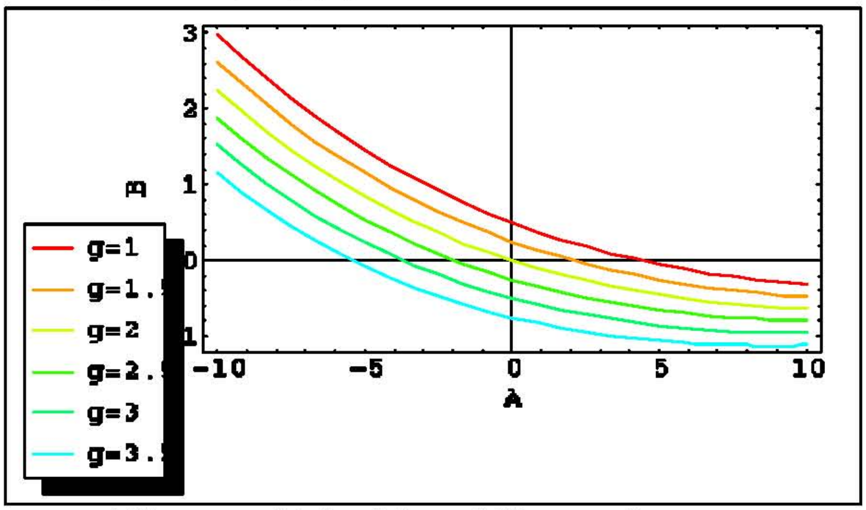

Fig. 2: Plot of the function $f(\lambda)$ with $\gamma = 1, 1.5, \ldots, 3.5$, B= $f(\lambda)$ and A= λ.

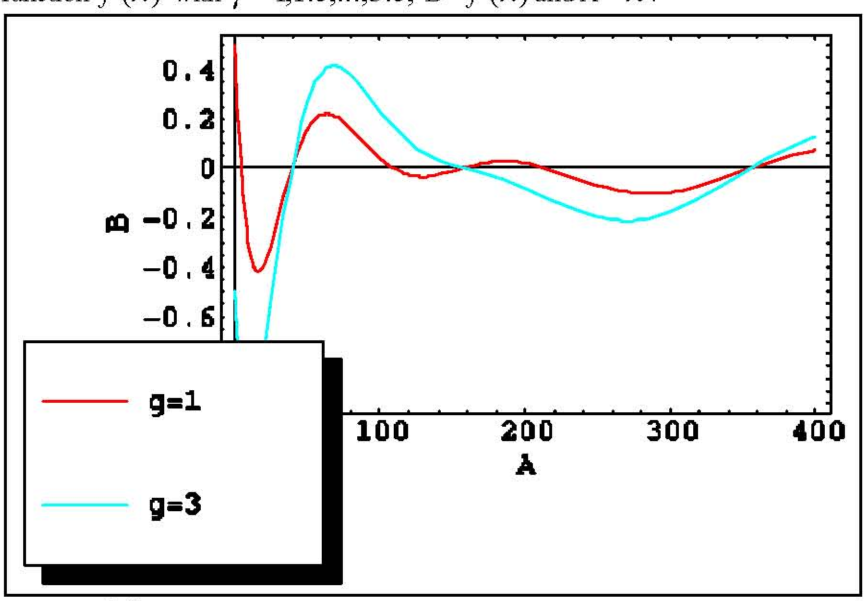

Fig. 3: Plot of the function $f(\lambda)$ with $\gamma = 1, 2,$ one bigger than 2 and another smaller than 2, B= $f(\lambda)$ and A= λ.

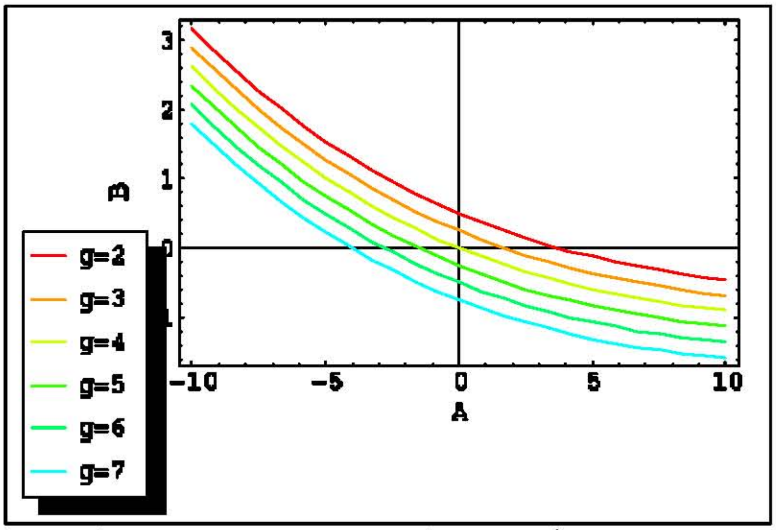

Fig. 4: Plot of the function $f(\lambda)$ with $\gamma = 1,2,\ldots,7$, B= $f(\lambda)$ and A= λ.

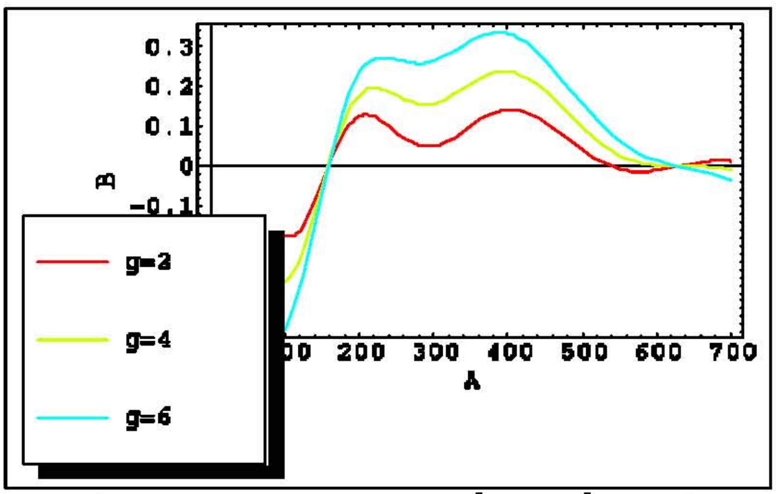

Fig. 5: Plot of the function $f(\lambda)$ with $\gamma = 2,4,6$ in the interval $[100,700]$, B= $f(\lambda)$ and A= λ.

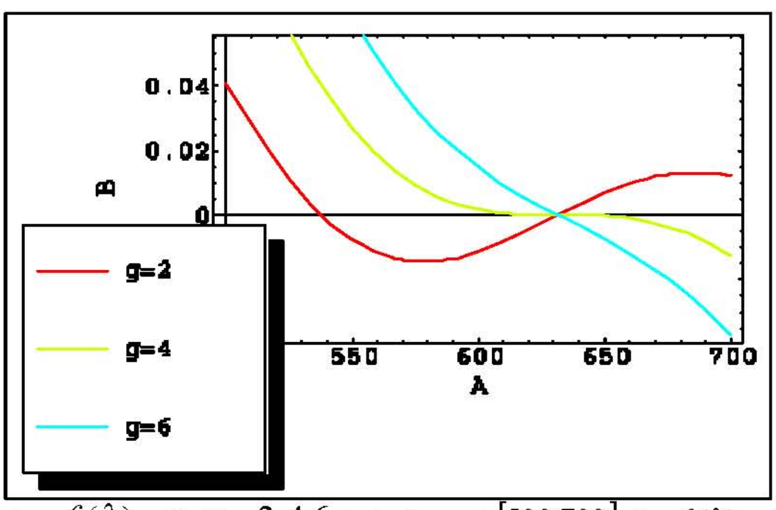

Fig. 6: Plot of the function $f(\lambda)$ with $\gamma = 2,4,6$ in the interval $[500,700]$, B= $f(\lambda)$ and A= λ.

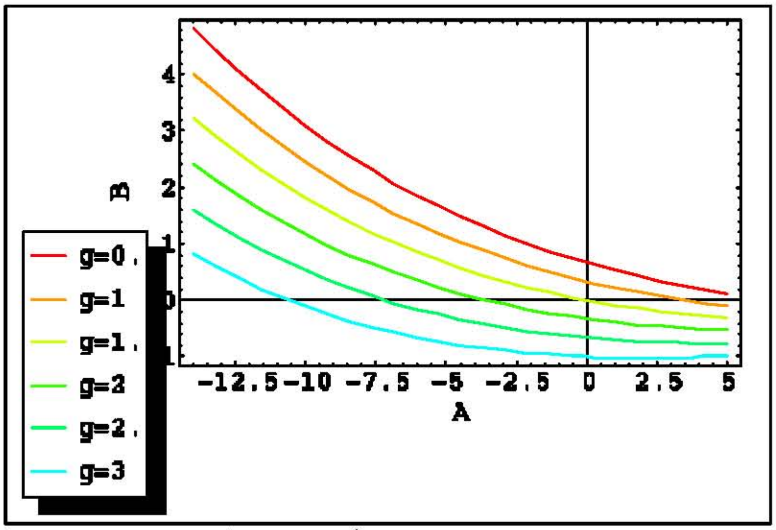

Fig. 7: $f(\lambda)$ with $\gamma = 0.5,1,\ldots,3$, B= $f(\lambda)$ and A= λ .

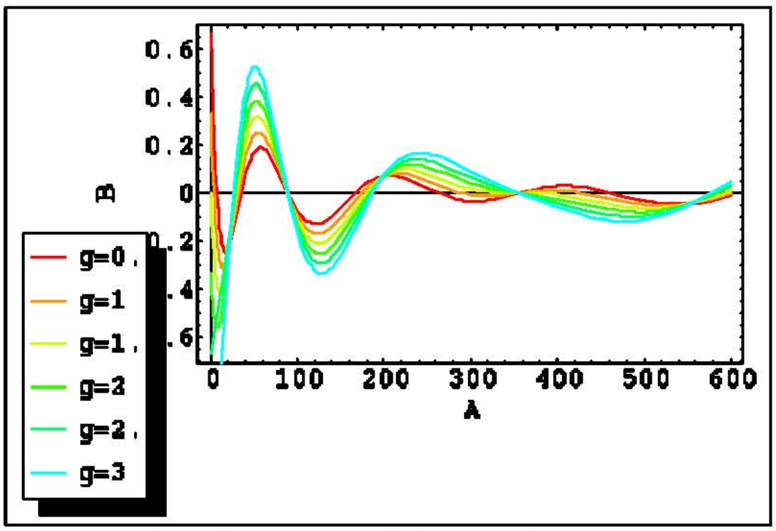

Fig. 8: $f(\lambda)$ with various γ in the interval [0,600], B= $f(\lambda)$ and A= λ .

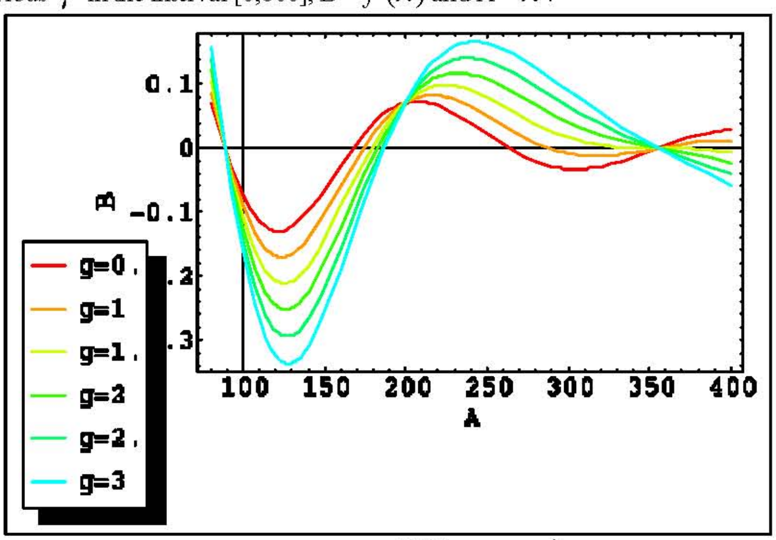

Fig. 9: $f(\lambda)$ with various γ in the interval [100,400], B= $f(\lambda)$ and A= λ .

3.3 Case 3: $\xi = \frac{1}{4}$

The behavior of the eigenvalues in the case of $\xi = \frac{1}{4}$ is similar to that of $\xi = \frac{1}{2}$. After implementing the VIM, the function *f* is plotted for different values of γ in Figures 4, 5 and 6. From Figure 4, it can be seen that for $\gamma > \frac{1}{\xi} = 4$ one negative eigenvalue exists. For $\gamma = \frac{1}{\xi} = 4, \lambda = 0$ is an eigenvalue and for $\gamma < \frac{1}{\xi} = 4$ there is no negative eigenvalue. Figures 5 and 6 are presented to show the behavior of the eigenvalues. In Table 2, the eigenvalues with different γ are presented.

In Figure 6, we have restricted the domain in the Figure 5 to see the behavior of eigenvalues more clearly.

3.4 Case 4: $\xi = \frac{2}{3}$

For $\xi = \frac{2}{3}$ after implementing the VIM, to show the location of the smallest eigenvalues, the function *f* is plotted for different values of γ in Figure 7. Again, it can be seen that for $\gamma > \frac{1}{\xi} = \frac{3}{2}$ one negative eigenvalue exists. For $\gamma = \frac{1}{\xi} = \frac{3}{2}, \lambda = 0$ is an eigenvalue and for $\gamma < \frac{1}{\xi} = \frac{3}{2}$ there is no negative eigenvalue. From Figures 8 and 9, the behavior of positive eigenvalues can be seen. Table 3 presents the first six eigenvalues with different γ.

In Figure 9, we have restricted the domain in the Figure 8 to see the behavior of eigenvalues more clearly.

CONCLUSIONS

A very efficient numerical method was presented to obtain multiple eigenvalues of a nonlocal Sturm-Liouville problem. The method was tested for different values of the parameters in the nonlocal boundary condition. We started the VIM algorithm with one initial approximation satisfying in the classical boundary condition and using the nonlocal boundary condition all eigenvalues were obtained. The implementation of the same algorithm proposed in this paper for finding multiple solutions of the nonlinear boundary value problems is our proposed direction for future research.

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