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Homotopy Analysis Method for Solving Nonlinear System of Equations

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Abstract: In this paper, we present an efficient numerical algorithm for solving system of nonlinear equations based on homotopy analysis method. Some numerical illustrations are given to show the efficiency of the algorithm. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

Key words: Homotopy analysis method . non-linear system equations . homotopy . convergence analysis . solution series

INTRODUCTION

Approximating the solutions of the system of linear and nonlinear equations has widespread applications in applied mathematics. In 1992, Liao [1] employed the basic ideas of homotopy to propose a general method for non-linear problems and modified it step by step [2-6]. This method has been successfully applied to solve many types of non-linear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called homotopy perturbation method (HPM), is provided by He [7-10]. In this article, HAM is applied to the solution of the system non-linear and the convergence of the method is considered under certain conditions.

ANALYSIS OF THE METHOD

Consider the system

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases} \qquad (1.1)$$

Let

$$A[u]=\begin{bmatrix} \dfrac{\partial f(x,y)}{\partial x} & \dfrac{\partial f(x,y)}{\partial y} \\ \dfrac{\partial g(x,y)}{\partial x} & \dfrac{\partial g(x,y)}{\partial y} \end{bmatrix}$$

$$b[u]=\begin{bmatrix} -f(x,y) \\ -g(x,y) \end{bmatrix}, \quad u=\begin{bmatrix} x \\ y \end{bmatrix}$$

Let

$$N[u]=A[u]u-b[u], \qquad L(u)=u$$

Suppose $q\in[0,1]$ denotes an embedding parameter, $\hbar\neq 0$ an auxiliary parameter, $H(u)\neq 0$ an auxiliary function and ℓ an auxiliary linear operator. We construct a homotopy

$$(1-q)\ell[v(q)-u_0]=\hbar H(u)q\{A[v(q)](v(q))-b[v(q)]\} \qquad (1.2)$$

where u_0 is the initial approximation of u and v(q) is an unknown function. It should be emphasized that one has great freedom to choose the initial guess value, the auxiliary linear operator, the auxiliary parameter $\hbar$ and the auxiliary function H(u). Obviously, when q = 0 and q = 1, it holds

$$v(0)=u_0, \qquad v(1)=u$$

respectively. When q increases from 0 to 1, v(q) varies from the initial guess u_0 to the solution u. Expanding v(q) in Taylor series with respect to the embedding parameter q, one has

$$v(q)=u_0+\sum_{m=1}^{\infty}u_m q^m \qquad (1.3)$$

where

$$u_m=\frac{1}{m!}\frac{d^m v(q)}{dq^m}\Big|_{q=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen that the series (1.3) converges at q=1, one has

$$u=u_0+\sum\nolimits_{m=1}^{\infty}u_m \qquad (1.4)$$

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which must be one of the solutions of (1.2), as proved by Liao [11]. It is very important to ensure the convergence of series (1.3) at q=1, otherwise, the series (1.4) has no meanings. As $\hbar = -1$ and H(u) =1, we obtained homotopy perturbation method [12] Setting $\ell \equiv L$ we have the high-order deformation equation [3]

$$L[u(m) - \chi_m u_{m-1}] = H(u)\hbar R_m(u_0, u_1, \cdots, u_m)$$

Where

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$$

$$R_m(u_0, u_1, \cdots, u_m) = \frac{1}{(m-1)!} \frac{d^m \{A[v(q)](v(q)) - b[v(q)]\}}{dq^m} |_{q=0}$$

According to rule of solution expression and the m-th order deformation equation (for m=1), the auxiliary function can be chosen as:

$$H(u) \equiv 1$$

Let us consider the initial approximation u_0, then we have

$$\begin{aligned} u_1 &= \hbar(A[u_0]u_0 - b[u_0]) \\ u_2 &= (\hbar A[u_1] + I)u_1 - \hbar b[u_1] \\ u_3 &= (\hbar A[u_2] + I)u_2 - \hbar b[u_2] \\ &\vdots \end{aligned}$$

And in general

$$u_{n+1} = (\hbar A[u_n] + I)u_n - \hbar b[u_n], \qquad n = 1, 2, \cdots$$

Hence, the solution can be of the form

$$u = u_0 + u_1 + u_3 + \cdots$$

NUMERICAL RESULTS

Here we illustrate the above mentioned methods with the help of three illustrative examples.

In this section, the absolute error has been calculated using this formula:

$$\text{Error} = \left| u_{exact} - u^N(\hbar) \right|$$

where $u^N(\hbar)$ is the approximate solution with N-replications.

Then any of the error function's entries is set to zero and its corresponding h is obtained. The lowest real value of these $\hbar$'s is considered as the approximate value of $\hbar$.

Example 1: Approximate the solution of the system nonlinear

$$\begin{cases} x^2 + y - 2 = 0 \\ xy + x - 2 = 0 \end{cases}$$

The true solution is $u^t = (1,1)$. For the given system we have

$$A[u]u = b[u]$$

where

$$A[u] = \begin{bmatrix} 2x & 1 \\ y+1 & x \end{bmatrix}, \quad b[u] = \begin{bmatrix} -x^2 - y + 2, \\ -xy - x + 2 \end{bmatrix}, \quad u = \begin{bmatrix} x \\ y \end{bmatrix}$$

From (1.3) and using four terms, we approximate the series solution when $\hbar = 0.039$ and $u_0^t = (1,1)$ are

$$u^3(h) \approx u_0 + u_1 + u_2 + u_3$$

Or

$$u^t \approx [0.9989259404, 0.9989259403]$$

Example 2: Solve the system nonlinear

$$\begin{cases} x^2 + y - 2 = 0 \\ x + y^2 - 2 = 0 \end{cases}$$

The true solution is $u^t = (1,1)$.

The four-term approximation when $\hbar = 0.05667$ and $u_0^t = (0.3, 0.3)$ are

$$u = \begin{bmatrix} 0.9999961969 \\ 0.9999961969 \end{bmatrix}$$

Example 3: Solve the system nonlinear

$$\begin{cases} x^2 + y - z - 2 = 0 \\ xy + z^2 - 3 = 0 \\ x^2 + y^2 - z^2 - 2 = 0 \end{cases}$$

The true solution is $u^t = (1,2,1)$.

The four-term approximation when $\hbar = 0.05234$ and $u_0^t = (0.349, 0.985, 0.2)$ are

$$u = \begin{bmatrix} 0.9999468233 \\ 1.999605754 \\ 0.9998632962 \end{bmatrix}$$

CONCLUSION

In this paper, we used homotopy analysis method to approximate the solution of system of nonlinear equations in terms of the subtraction of the coefficient and unit matrices. The above examples illustrate the strength and reliability of the method presented in this paper and reveals that it is very simple and effective. The obtained solutions, in comparison with exact solutions admit a remarkable accuracy. Results indicate that the convergence rate is very fast and lower approximations can also achieve high accuracy. Up to now, there are no rigorous theories to direct us to choose the initial approximations, auxiliary linear operators, auxiliary functions and auxiliary parameter $\hbar$.

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