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Modified Homotopy Analysis Method for Solving of Linear System Equations

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Abstract: In this paper, we present an efficient numerical algorithm for solving system of linear equations based on modified homotopy analysis method. Some numerical illustrations are given to show the efficiency of the algorithm. The homotopy analysis method contains the auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series.

AMS subject classification: 15A06

Key words: Modified homotopy analysis method . linear system equations . homotopy . convergence analysis . solution series

INTRODUCTION

Approximating the solutions of the system of linear and nonlinear equations has widespread applications in applied mathematics. In 1992, Liao [6] employed the basic ideas of homotopy to propose a general method for nonlinear problems and modified it step by step [7-9, 11, 12]. This method has been successfully applied to solve many types of nonlinear problems. Following Liao, an analytic approach based on the same theory in 1998, which is so called homotopy perturbation method (HPM), is provided by He [1-3, 5]. In this article, MHAM is applied to the solution of the system $Ax = b$ and the convergence of the method is considered under certain conditions.

DESCRIPTION OF THE METHOD

Consider the system

$$Ax = b \qquad (1)$$

where

$$A=[a_{ij}],\quad x=[x_j],\quad b=[b_i],\quad i=1,2,\ldots,\quad j=1,2,\ldots$$

Let

$$N(u)=Au-b,\qquad L(u)=u$$

Suppose $q\in[0,1]$ denotes an embedding parameter, $\hbar_i \neq 0$ an auxiliary parameter and $\mathcal{L}$ an auxiliary linear operator. We construct the zero-order deformation equation [8]

$$(1-q)\mathcal{L}[v_i(q)-u_{0,i-1}] = \hbar_i q\{A(v_i(q))-b\},\qquad i=1,2,\cdots \qquad (2)$$

where $u_{0,i-1}$ is the initial approximation of $u_{0,i}$ and $v_i(q)$ is a unknown function. It should be emphasized that one has great freedom to choose the initial guess value, the auxiliary linear operator and the auxiliary parameter $\hbar$. Obviously, when q = 0 and q = 1, it holds

$$v_i(0)=u_{0,i-1},\ v_i(1)=u_{0,i},\qquad i=1,2,\cdots$$

respectively. When q increases from 0 to 0, $v_i(q)$ varies from the initial guess $u_{0,i-1}$ to the solution $u_{0,i}$. Expanding $v_i(q)$ in Taylor series with respect to the embedding parameter q, one has

$$v_i(q)=u_{0,i-1}+\sum_{m=1}^{\infty}u_{m,i-1}q^m,\qquad i=1,2,\cdots \qquad (3)$$

where

$$u_{m,i-1}=\frac{1}{m!}\frac{d^m v_i(q)}{dq^m}\Big|_{q=0},\qquad i=1,2,\cdots$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$ and the auxiliary function are so properly chosen that the series (5) converges at q = 1, one has

$$u_{0,i}=u_{0,i-1}+\sum_{m=1}^{\infty}u_{m,i-1}q^m,\qquad i=1,2,\cdots \qquad (4)$$

which must be one of the solutions of (1), as proved by Liao [10]. It is very important to ensure

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the convergence of series (5) at q = 1, otherwise, the series (4) has no meanings. Setting $\mathcal{L} \equiv L$, we have the high-order deformation equation [8]

$$L[u_{m,i-1} - \chi_m u_{m-1,i-1}] = \hbar_i R_{m,i-1}(u_{0,i-1}, u_{1,i-1}, \cdots, u_{m,i-1}), i=1,2,\cdots$$

where

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$$

$$R_{m,i-1}(u_{0,i-1}, u_{1,i-1}, \cdots, u_{m,i-1}) = \frac{1}{(m-1)!}\frac{d^m\{A(v_i(q)) - b\}}{dq^m}\Big|_{q=0}, \quad i=1,2,\cdots$$

if we take $u_{0,0} = 0$ then we have

$$\begin{aligned} u_{1,i-1} &= (\hbar_i A)u_{0,i-1} - \hbar_i b, \quad i=1,2,\cdots \\ u_{2,i-1} &= (\hbar_i A + I)u_{1,i-1} \quad i=1,2,\cdots \\ u_{3,i-1} &= (\hbar_i A + I)u_{2,i-1} \quad i=1,2,\cdots \\ &\vdots \end{aligned}$$

and in general

$$u_{n+1,i-1} = (\hbar_i A + I)u_{n,i-1}, \quad n=1,2,\cdots, \quad i=1,2,\cdots$$

hence, the solution can be of the form

$$u_{0,i} = u_{0,i-1} + u_{1,i-1} + u_{2,i-1} + \cdots, \qquad i=1,2,\cdots$$

or

$$u_{0,i} = -\hbar_i[I + (\hbar_i A + I) + (\hbar_i A + I)^2 + \cdots]b, \ \ i=1,2,\cdots \qquad (5)$$

Theorem 1. The sequence

$$u^{[m]} = [-\hbar\sum_{k=0}^{m}(\hbar A + I)^k]b$$

is a Chauchy sequence if

$$\|\hbar A + I\| < 1$$

Proof: we must show that

$$\lim_{m\to\infty}\| u^{[m+p]} - u^{[m]} \| = 0$$

so for showing this we can write

$$u^{[m+p]} - u^{[m]} = [-\hbar\sum_{k=1}^{p}(\hbar A + I)^{m+k}]b$$

or

$$\| u^{[m+p]} - u^{[m]} \| \le \| -\hbar b \| \sum_{k=1}^{p} \| (\hbar A + I)^{m+k} \|$$

let $\gamma = \|(\hbar A + I)^{m+k}\|$, then

$$\| u^{[m+p]} - u^{[m]} \| \le \| -\hbar b \| \gamma^m \sum_{k=1}^{p}\gamma^k \le \| -\hbar b \| (\frac{\gamma^p - 1}{\gamma - 1})\gamma^m$$

now if $\gamma<1$, then we have

$$\lim_{m\to\infty}\| u^{[m+p]} - u^{[m]} \| \le \| -\hbar b \| (\frac{\gamma^p - 1}{\gamma - 1})(\lim_{m\to\infty}\gamma^m)$$

hence,we obtain

$$\lim_{m\to\infty}\| u^{[m+p]} - u^{[m]} \| = 0$$

which completes the proof.

NUMERICAL RESULTS

Here we illustrate the above mentioned methods with the help of eight illustrative examples.

Example 1: Approximate the solution of the system

$$\begin{cases} 4x + y - z = 7 \\ -x + 6y + 2z = 9 \\ y - 3z = 5 \end{cases}$$

The true solution is u^t = (1,2,-1). For the given system we have

$$Au = b$$

where

$$A = \begin{bmatrix} 4 & 1 & -1 \\ -1 & 6 & 2 \\ 0 & 1 & -3 \end{bmatrix}, \qquad \begin{cases} b^t = [7,9,5] \\ \\ u' = [x,y,z] \end{cases}$$

since A is diagonally dominated we write the new system as follows

$$Bu = q$$

in which

$$B = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{6} & 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}, \quad q^t = \left[\frac{7}{4}, \frac{9}{6}, -\frac{5}{3}\right]$$

Table 1: Numerical results for example 1

i	h_i	$u^t_{0,i}$	$\lvert u_{exact} - u^t_{0,i} \rvert$
1	-0.79	(0.9873314426, 1.000487261, -0.990487625)	(1.3E-2,4.9E-4, 9.5E-3)
2	-0.79	(0.9998966527, 1.000066902, -0.999948344)	(1.1E-4, 6.7E-5, 5.2E-5)
3	-0.75	(0.9999999299, 1.000000805, -1.000000339)	(7.1E-8, 8.1E-7, 3.4E-7)

Table 2: Numerical results for example 2

i	h_i	$u^t_{0,i}$	$\lvert u_{exact} - u^t_{0,i} \rvert$
1	-2.18	(1.021480884, 0.9991607836, 0.9991607836)	(2.1E-2,8.4E-4, 8.4E-4)
2	-2.18	(1.000036759, 0.9999281262, 0.9999281262)	(3.7E-5,7.2E-5, 7.2E-5)
3	-2.12	(1.000001618, 0.9999994288, 0.9999994288)	(1.6E-6,5.7E-7, 5.7E-7)
4	-2.36	(1.000000001, 0.9999999931, 0.9999999931)	(1.0E-9,6.9E-9, 6.9E-9)

from (5) we have

$$u_{0,i} = -\hbar_i\,[I + (\hbar_i A + I) + (\hbar_i A + I)^2 + \cdots]q, \qquad i = 1,2,3$$

and using four terms, we approximate the series solution

$$u_{0,i} \approx u_{0,i-1} + u_{1,i-1} + u_{2,i-1} + u_{3,i-1}, \qquad i = 1,2,3$$

Table 1 shows absolute errors of numerical results calculated according to the presented method.

Example 2: Solve the system $Au = b$, where

$$A = \begin{bmatrix} 0.5 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}, \quad b = \begin{bmatrix} 1.2 \\ 0.5 \\ 0.5 \end{bmatrix}$$

The true solution is $u^t = (1,1,1)$. The ten terms, we approximate the series solution

$$u_{0,i} \approx u_{0,i-1} + \sum_{j=1}^{9} u_{j,i-1}, \qquad i = 1,2,3,4$$

Table 2 shows absolute errors of numerical results calculated according to the presented method.

CONCLUSION

In this paper, we used modified homotopy analysis method to approximate the solution of system of linear equations in terms of the subtraction of the coefficient and unit matrices. Solved problems show the convergence of the method increases as the coefficient matrix becomes more strictly diagonally dominated.

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