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An Efficient Laplace Decomposition Algorithm for Fourth Order Parabolic Partial Differential Equations with Variable Coefficients

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Abstract: The purpose of this article is to introduce a new algorithm, namely Laplace Decomposition Algorithm (LDA) for fourth order parabolic partial differential equations with variable coefficients. This equation arises in the transverse vibration problems. The proposed iterative scheme finds the solution without any discretization, linearization and other restrictive assumptions. Some applications are given to verify the reliability and efficiency of the method. This new algorithm provides us with a convenient way to find exact solution with less computation.

Key words: Laplace decomposition algorithm . fourth order parabolic partial differential equations . exact solution

INTRODUCTION

The decomposition method has been shown to solve [1-5] efficiently, easily and accurately a large class of linear and nonlinear ordinary, partial, deterministic or stochastic differentia equations. The method is very we suited to physical problems since it does not require unnecessary linearization, perturbation and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously.

The Laplace decomposition method was first proposed by Khuri [6, 7] with coupling of standard Adomian decomposition method and Laplace transform for solving nonlinear differentia equations and Bratu's problem. There is no need of linearization, discretization and large computational work. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation. Recently Majid *et al.* have been introduced various modifications in Laplace decomposition to idea with nonlinear behaviors of the physical models [8-11]. It is worth mentioning that the proposed method is an elegant combination of Laplace transform and decomposition method. The advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for nonlinear equations.

LAPLACE DECOMPOSITION ALGORITHM

Consider the fourth order parabolic linear partial differentia equation with variable coefficients of the form

$$u_{tt} + \xi(x)u_{xxxx} = g(x,t) \qquad (2.1)$$

where $\xi(x)$ is a variable coefficient. The initial conditions with Eq. (2.1) are of the form

$$u(x,0) = f(x), \quad u_t(x,0) = h(x) \qquad (2.2)$$

and the boundary conditions associated with Eq. (2.1) are

$$\begin{aligned} u(\alpha,t) &= k(t), \quad u(\beta,t) = l(t), \qquad t>0 \\ u_{xx}(\alpha,t) &= b(t), \quad u_{xx}(\beta,t) = d(t), \quad t>0 \end{aligned} \qquad (2.3)$$

By applying Laplace transform on both sides of Eq. (2.1), we get

$$£\left[u_{tt} + \xi(x)u_{xxxx} = g(x,t)\right] \qquad (2.4)$$

Using the differentia property of Laplace transform, we have

$$s^2 £\left[u(x,t)\right] - su(x,0) - u_t(x,0) + £\left[\xi(x)u_{xxxx}(x,t)\right] = £\left[g(x,t)\right]$$

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$$\pounds[u(x,t)] = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} + \pounds[g(x,t)] - \pounds[\xi(x)u_{xxxx}(x,t)] \quad (2.5)$$

Using given initial conditions (2.2) yields

$$\pounds[u(x,t)] = \frac{f(x)}{s} + \frac{h(x)}{s^2} + \frac{1}{s^2}\pounds[g(x,t)] - \frac{1}{s^2}\pounds[\xi(x)u_{xxxx}(x,t)] \quad (2.6)$$

Operating inverse Laplace transform on both sides of Eq. (2.6) we get

$$u(x,t) = G(x,t) - \pounds^{-1}\left[\frac{1}{s^2}\left[\pounds[\xi(x)u_{xxxx}(x,t)]\right]\right] \quad (2.7)$$

The Laplace decomposition algorithm assumes the solution u can be expanded into infinite series as

$$u = \sum_{m=0}^{\infty} u_m \quad (2.8)$$

By substituting Eq. (2.8) in Eq. (2.7), the solution can be written as

$$\sum_{m=0}^{\infty} u_m(x,t) = G(x,t) - \pounds^{-1}\left[\frac{1}{s^2}\left[\pounds\left[\xi(x)\sum_{m=0}^{\infty} u_{mxxxx}(x,t)\right]\right]\right] \quad (2.9)$$

In general, the recursive relation is given by

$$u_0(x,t) = G(x,t)$$

$$u_{m+1}(x,t) = -\pounds^{-1}\left[\frac{1}{s^2}\left[\pounds\left[\xi(x)\sum_{m=0}^{\infty} u_{mxxxx}(x,t)\right]\right]\right], m \geq 0 \quad (2.10)$$

CASE STUDY

In this section, some examples are given in order to demonstrate the effectiveness of LDA for fourth order parabolic partial differentia equations.

Homogeneous fourth order parabolic partial differentia equation: Let us consider fourth order homogeneous parabolic partial differentia equation

$$u_{tt} + \left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxxx} = 0, \qquad \frac{1}{2} < x < 1, t > 0 \quad (3.1)$$

with initial and boundary associated with Eq. (3.1) are

$$u(x,0) = 0. \quad u_t(x,0) = 1 + \frac{x^5}{120}, \qquad \frac{1}{2} < x < 1 \quad (3.2)$$

$$u(\frac{1}{2},t) = \left(1 + \frac{(\frac{1}{2})^5}{120}\right)\sin t, \quad u(1,0) = \frac{121}{120}\sin t, \quad t > 0 \quad (3.3)$$

$$u_{xx}(\frac{1}{2},t) = \frac{1}{6}\left(\frac{1}{2}\right)^3 \sin t, \qquad u_{xx}(1,0) = \frac{1}{6}\sin t, \quad t > 0 \quad (3.4)$$

Applying Laplace transform algorithm we have

$$s^2 u(x,s) - su(x,0) - u_t(x,0) = -\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxxx}\right]$$

$$u(x,s) = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} - \frac{1}{s^2}\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxxx}\right] \quad (3.5)$$

Using given initial conditions Eq. (3.2) become

$$u(x,s) = \frac{\left(1 + \frac{x^5}{120}\right)}{s^2} - \frac{1}{s^2}\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxxx}\right] \quad (3.6)$$

Applying inverse Laplace transform to Eq. (3.6) yields

$$u(x,t) = \left(1 + \frac{x^5}{120}\right)t - \pounds^{-1}\left[\frac{1}{s^2}\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{xxxxx}\right]\right] \quad (3.7)$$

The Laplace Decomposition Method (LDA) [6, 7] assumes a series solution of the function u(x,t) is given by

$$u = \sum_{n=0}^{\infty} u_n(x,t) \quad (3.8)$$

Using Eq. (3.8) into Eq. (3.7) we get

$$\sum_{n=0}^{\infty} u_n(x,t) = \left(1 + \frac{x^5}{120}\right)t - \pounds^{-1}\left[\frac{1}{s^2}\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)\sum_{n=0}^{\infty} u_{xxxxx}\right]\right] \quad (3.9)$$

From Eq. (3.9), our required recursive relation is given below

$$u_0(x,t) = \left(1 + \frac{x^5}{120}\right)t \quad (3.10)$$

$$u_{n+1}(x,t) = -\pounds^{-1}\left[\frac{1}{s^2}\pounds\left[\left(\frac{1}{x} + \frac{x^4}{120}\right)u_{nxxxxx}\right]\right], \; n \geq 0 \quad (3.11)$$

The first few components of $u_n(x,t)$ by using recursive relation (3.11) as follows immediately

$$u_1(x,t) = -\left(1+\frac{x^5}{120}\right)\frac{t^3}{3!} \quad (3.12)$$

$$u_2(x,t) = \left(1+\frac{x^5}{120}\right)\frac{t^5}{5!} \quad (3.13)$$

$$\vdots$$

The solution in a series form is

$$u(x,t) = \left(1+\frac{x^5}{120}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-...\right) \quad (3.14)$$

The closed form solution is given by

$$u(x,t) = \left(1+\frac{x^5}{5!}\right)\sin t \quad (3.15)$$

It is worth mentioning that we use only initial conditions to get above solution. The obtained solution satisfies the four given boundary conditions as we which are not used in finding the solution.

Inhomogeneous fourth order parabolic partial differentia equation: Let us consider fourth order inhomogeneous parabolic partial differentia equation

$$u_{tt} + (1+x)u_{xxxxx} = \left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t, 0<x<1, t>0 \quad (3.16)$$

with initial and boundary associated with Eq. (3.16) are

$$u(x,0) = \frac{6}{7!}x^7, \quad u_t(x,0) = 0, \quad 0<x<1 \quad (3.17)$$

$$u(0,t) = 0, \ u(1,0) = \frac{6}{7!}\cos t, \quad t>0 \quad (3.18)$$

$$u_{xx}(0,t) = 0, \ u_{xx}(1,0) = \frac{1}{20}\cos t, \quad t>0 \quad (3.19)$$

Applying Laplace transform algorithm we have

$$s^2 u(x,s) - su(x,0) - u_t(x,0) = £\left[\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t\right] - £\left[(1+x)u_{xxxxx}\right]$$

$$u(x,s) = \frac{u(x,0)}{s} + \frac{u_t(x,0)}{s^2} + £\left[\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t\right] - \frac{1}{s^2}£\left[(1+x)u_{xxxxx}\right] \quad (3.20)$$

Using given initial conditions Eq. (3.17) becomes

$$u(x,s) = \frac{\frac{6}{7!}x^7}{s} + \frac{1}{s^2}£\left[\left(x^4+x^3-\frac{6}{7!}x^7\right)\cos t\right] - \frac{1}{s^2}£\left[(1+x)u_{xxxxx}\right] \quad (3.21)$$

Applying inverse Laplace transform to Eq. (3.21) we get

$$u(x,t) = \frac{6}{7!}x^7 + \left(x^4+x^3-\frac{6}{7!}x^7\right)(1-\cos t) - £^{-1}\left[\frac{1}{s^2}£\left[(1+x)u_{xxxxx}\right]\right] \quad (3.22)$$

The Laplace Decomposition Method (LDA) assumes a series solution of the function u(x,t) is given by

$$u = \sum_{n=0}^{\infty} u_n(x,t) \quad (3.23)$$

Using Eq. (3.23) into Eq. (3.22) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{6}{7!}x^7 + \left(x^4+x^3-\frac{6}{7!}x^7\right)(1-\cos t) - £^{-1}\left[\frac{1}{s^2}£\left[(1+x)\sum_{n=0}^{\infty} u_{xxxxx}\right]\right] \quad (3.24)$$

From Eq. (3.24), our required recursive relation is given below

$$u_0(x,t) = \frac{6}{7!}x^7 + \left(x^4 + x^3 - \frac{6}{7!}x^7\right)(1 - \cos t) \qquad (3.25)$$

$$u_{n+1}(x,t) = -£^{-1}\left[\frac{1}{s^2}£\left[(1+x)\,u_{nxxxxx}\right]\right],\ n \geq 0 \qquad (3.26)$$

As we proceed, we have noise terms appears in zeroth and first order solutions and with the advantage of noise terms phenomena we get

$$u(x,t) = \frac{6}{7!}x^7 \qquad (3.27)$$

which is the exact solution of Eq. (3.16).

CONCLUSION

In this work, we have proposed Laplace decomposition algorithm. In examples, we showed that LDA is applied to fourth order parabolic partial differential equations with variable coefficients and we obtained exact solution of the understudy problem equations with a less computational work involved in it. The obtained results in these examples indicate that LDA is feasible and effective.

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