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A Fractional Model of Impurity Concentration and Its Approximate Solution

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Abstract: This paper proposed a fractional model for the flow rate and characteristic of the impurity of order α, $\beta(0<\alpha, \beta\leq 1)$ respectively, which describes one dimensional dynamical flows of electrically conducting fluid. In this model fractional derivatives are described in the Caputo sense. The beauty of the paper is residual analysis which shows that our approximate solution converges very rapidly to the exact solution. Numerical results show that the HPM is easy to implement and accurate when applied to the time fractional partial differential equation. Numerical results are presented graphically.

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INTRODUCTION

In the past few decades, fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearances in various applications in fluid mechanics, viscoelasticity, biology, physics, electrical network, control theory of dynamical systems, chemical physics, optics and signal processing, as they can be modelled by linear and non-linear fractional order differential equations. The book by Oldham and Spanier [1] has played a key role in the development of the subject. Some fundamental results related to solving fractional differential equations may be found in Miller and Rose [2], Podlubny [3], Kilbas and Srivastava [4], Diethelm and Ford [5], Diethelm [6].

We consider the simplest dissipative one dimensional model of a medium of viscous heat conducting liquid with pressure taking the transfer of passive impurity into account, which has no inverse effect on the dynamics, are described by the following system of nonlinear partial differential equations

$$\begin{cases} \dfrac{\partial w}{\partial t} + w\dfrac{\partial w}{\partial x} = \nu\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\gamma}{2}\dfrac{\partial c^2}{\partial x}, \ [7-8] \\ \dfrac{\partial c}{\partial t} + \dfrac{\partial (wc)}{\partial x} = \chi\dfrac{\partial^2 c}{\partial x^2} \end{cases} \qquad (1)$$

where $w = w(x,t)$ is the flow rate and $c = c(x,t)$ is a characteristic of the impurity (the temperature in the case of heat transfer or the concentration in the case of mass transfer in a two component liquid). Here ν and χ are the coefficient of viscosity and thermal diffusivity (or the diffusion coefficient), the non-dimensional ratio of which is the Prandtl number (or Schmidt) $Pr = \nu/\chi$.

The objective of this paper is to extend the application of the homotopy perturbation method (HPM) to obtain analytic and approximate solutions of the time fractional system of nonlinear partial differential equations (1) of the flow rate and characteristic of the impurity. The homotopy perturbation method was first proposed by the Chinese mathematician He [9] and was successfully applied to solve nonlinear wave equations by He [10]. The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1, when $p = 0$, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p gradually increases to 1, the system goes through a sequence of deformations, the solution for each of which is close to that of the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Nonlinear partial differential equations have many applications in various fields of science and engineering such as fluid

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mechanics, thermodynamics, mass and heat transfer, micro electro mechanics system etc. It is difficult to handle nonlinear part of these equations. Many researchers [11-17] applied HPM to find the solution of various nonlinear fractional ordinary and partial differential equations describing various physical and engineering models. Recently, Shidfar *et al*. [18] have solved the nonlinear system of partial differential equation arising in magnetic field by using homotopy perturbation method. This [18] motivated us for the present work.

PRELIMINARIES AND NOTATIONS

In this section, we give some definitions and properties of the fractional calculus which are used further in this paper.

Definition 2.1: A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu\in R$, if there exists a real number $p(>\mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) = C[0,\infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)}\in C_\mu$, $m\in N$.

Definition 2.2: The Riemann-Liouville fractional integral operator (j^α) of order $\alpha\geq 0$, of the function $f\in C_\mu$, $\mu\geq -1$, is defind as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,\ \ \alpha>0,\ x>0$$

$$J^0 f(x) = f(x)$$

Properties of the operator (j^α), can be found in [1-4], we mentation only the following. For $f\in C_\mu$, $\mu\geq -1$ and $\gamma\geq -1$:

(1) $J^\alpha J^\beta f(x) = J^{\alpha+\beta}f(x)$

(2) $\left(J^\alpha J^\beta\right)f(x) = \left(J^\beta J^\alpha\right)f(x)$

(3) $J^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}x^{\gamma+\alpha}$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work in the theory of viscoelasticity [19].

Definition 2.3: The fractional derivatives (D^α) of $f(x)$ in the Caputo's sense is defined as:

$$D^\alpha f(x) = J^{m-\alpha}D^m f(x) = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha+1-m}}dt,\ \ \alpha>0,\ x>0 \quad (2)$$

for $m-1<\mathrm{Re}(\alpha)\leq m,\ m\in N,\ f\in C_{-1}^m$

The following are two basic properties of the Caputo's fractional derivative:

Lemma 2.1: If $m-1<\alpha\leq m, m\in N$ and $f\in C_\mu^n$, $\mu\geq -1$, then

$$(D^\alpha J^\alpha)f(x) = f(x)$$

$$(J^\alpha D^\alpha)f(x) = f(x) - \sum_{i=0}^{m-1} f^i(0^+)\frac{x^i}{i!}$$

The Caputo fractional derivatives are considered here because it allows traditional initial conditions to be included in the formulation of the problem.

Definition 2.4: For m to be the smallest integer that exceed α, the Caputo time fractional derivatives operator of $\alpha>0$ is defined as:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m} & \text{for } m-1<\alpha<m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m\in N \end{cases} \quad (3)$$

METHOD OF SOLUTION

We consider the following fractional version of the standard nonlinear partial differential equations of the flow rate and characteristic of the impurity

$$\begin{cases} \dfrac{\partial^\alpha w}{\partial t^\alpha} + w\dfrac{\partial w}{\partial x} = -\dfrac{\gamma}{2}\dfrac{\partial c^2}{\partial x} + \nu\dfrac{\partial^2 w}{\partial x^2}, & t>0,\ \ 0<\alpha\leq 1 \\ \dfrac{\partial^\beta c}{\partial t^\beta} + \dfrac{\partial (wc)}{\partial x} = \chi\dfrac{\partial^2 c}{\partial x^2}, & t>0,\ \ 0<\beta\leq 1 \end{cases} \quad (4)$$

with initial conditions $w(x,0) = f(x)$ and $c(x,0) = g(x)$. We construct the following homotopy

$$\begin{cases} D_t^{\alpha} w = p[\nu D_{xx} w - w D_x w - \frac{\gamma}{2} D_x c^2], & 0 < \alpha \le 1 \\ D_t^{\beta} c = p[\chi D_{xx} c - D_x (wc)], & 0 < \beta \le 1 \end{cases} \quad (5)$$

where

$$D_t^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}},\ D_t^{\beta} = \frac{\partial^{\beta}}{\partial t^{\beta}}$$

$$D_x = \frac{\partial}{\partial x} \text{ and } D_x^2 = \frac{\partial^2}{\partial x^2}$$

to solve the system of PDEs (4).

Now applying the classical perturbation technique, we can assume that the solutions, $w(x,t)$ and $c(x,t)$ of Eqs. (5) can be expressed as a power series in p as follows

$$w(x,t) = w_0(x,t) + p w_1(x,t) + p^2 w_2(x,t) + p^3 w_3(x,t) + \dots \quad (6)$$

$$c(x,t) = c_0(x,t) + p c_1(x,t) + p^2 c_2(x,t) + p^3 c_3(x,t) + \dots \quad (7)$$

Substituting (6)-(7) into (5) and equating the coefficients of like powers of p, we get the following set of differential equations

$$p^0 : \begin{cases} D_t^{\alpha} w_0(x,t) = 0 \\ D_t^{\beta} c_0(x,t) = 0 \end{cases} \quad (8)$$

$$p^1 : \begin{cases} D_t^{\alpha} w_1 = \nu D_{xx} w_0 - w_0 D_x w_0 - \frac{\gamma}{2} D_x c_0^2 \\ D_t^{\beta} c_1 = \chi D_{xx} c_0 - D_x (w_0 c_0) \end{cases} \quad (9)$$

$$p^2 : \begin{cases} D_t^{\alpha} w_2 = \nu D_{xx} w_1 - (w_0 D_x w_1 + w_1 D_x w_0) - \frac{\gamma}{2} D_x (2 c_0 c_1) \\ D_t^{\beta} c_2 = \chi D_{xx} c_1 - D_x (w_0 c_1 + w_1 c_0) \end{cases} \quad (10)$$

$$p^3 : \begin{cases} D_t^{\alpha} w_3 = \nu D_{xx} w_2 - (w_0 D_x w_2 + w_1 D_x w_1 + w_2 D_x w_0) \\ \qquad - \frac{\gamma}{2} D_x (c_1^2 + 2 c_0 c_2) \\ D_t^{\beta} c_3 = \chi D_{xx} c_2 - D_x (w_0 c_2 + w_1 c_1 + w_2 c_0) \end{cases} \quad (11)$$

and so on. The above system of nonlinear equations can be easily solved by applying the operator J_t^{α} to (8)-(11) giving the various components $w_n(x,t)$ and $c_n(x,t)$, thus enabling the series solution to be entirely determined. The solutions $w(x,t)$ and $c(x,t)$ are given by

$$w(x,t) = \lim_{N\to\infty} \phi_N(x,t)$$

$$c(x,t) = \lim_{N\to\infty} \psi_N(x,t) \quad (12)$$

where

$$\phi_N(x,t) = \sum_{n=0}^{N-1} w_n(x,t)$$

and

$$\psi_N(x,t) = \sum_{n=0}^{N-1} c_n(x,t)$$

The series solution converges very rapidly. The rapid convergence means only few terms are required to get the approximate solution.

ILLUSTRATIVE EXAMPLES

In this section, we illustrate the solution and show the capability of the method. Two numerical examples are considered. As the exact solutions of the two examples are not known, to establish the accuracy of the proposed method, we define residuals

$$R_m(w) = |w_m - w_{m-1}|$$

and

$$R_m(c) = |c_m - c_{m-1}|$$

and show that $R_m(w)$ and $R_m(c)$ are monotonically decreasing with m. In both example, we have taken all kinetic coefficient same i.e. $\nu = \chi = \gamma = 1$ for all graphical results.

Example 1: We consider the model described by system (4) with initial conditions as (Shidfar *et al*. [18])

$$w(x,0) = \frac{(5x+7)}{12}$$

and

$$c(x,0) = \frac{(1-x)}{12} \ [16] \quad (13)$$

Applying the operator J_t^{α} (the inverse operator of the Caputo derivative D_t^{α}) on both sides of Eqs. (9)-(11), we obtain

$$w_0(x,t) = w(x,0) = \frac{(5x+7)}{12}$$

$$c_0(x,t) = c(x,0) = \frac{(1-x)}{12}$$

$$w_1(x,t) = -\frac{(13x+17)}{72}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$c_1(x,t) = \frac{(5x+1)}{72}\frac{t^{\beta}}{\Gamma(\beta+1)}$$

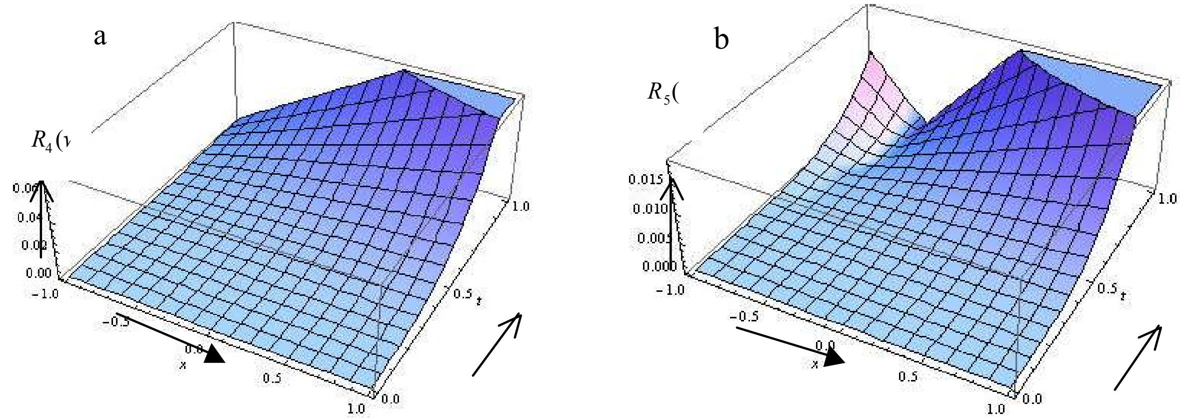

Fig. 1: Plots of residuals $R_4(w)$ and $R_5(w)$ for $\alpha = \beta = 1$

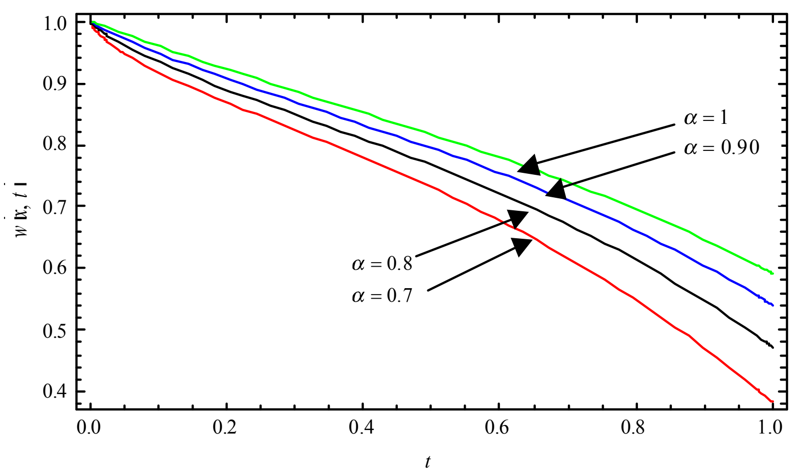

Fig. 2: Plot of w(x,t) vs. time t at x = 1 and $\beta = 1$ for different value of α

$$w_2(x,t) = \left(\frac{5x(25+\gamma)+175+\gamma}{864}\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{(5x-2)\gamma}{432}\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$c_2(x,t) = -\frac{5(5x+4)}{432}\frac{t^{2\beta}}{\Gamma(2\beta+1)} - \left(\frac{x(25+\gamma)+5-\gamma}{864}\right)\frac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$$

$$w_3(x,t) = -\left[\frac{5(5x(25+\gamma)+175+4\gamma)}{5184} + \frac{(25+\gamma)\left(x(25+\gamma)+35-\gamma\right)\Gamma(2\alpha+1)}{20736(\Gamma(\alpha+1))^2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{\gamma\left(x(75+\gamma)+15-\gamma\right)}{5184}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}$$
$$-\frac{5\gamma}{5184}\left[(10x+1)+\frac{(5x+1)\Gamma(2\beta+1)}{(\Gamma(\beta+1))^2}\right]\frac{t^{\alpha+2\beta}}{\Gamma(\alpha+2\beta+1)},\ldots$$

$$c_3(x,t) = \frac{25(10x+11)}{5184}\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \left[\frac{\left(5x(25+3\gamma)+100-6\gamma\right)}{5184} + \frac{\left(5x(25+\gamma)+100-2\gamma\right)\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\alpha+2\beta+1)}\right] + \frac{\left(5x(25+\gamma)+25-2\gamma\right)}{5184}\frac{t^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)},\ldots$$

The rest of components of $w_n(x,t)$ and $c_n(x,t)$ can be obtained similarly and summing them up give the series solutions. We have discussed the residual

$$R_n(w) = \left|w_n(x,t) - w_{n-1}(x,t)\right| \text{ and } R_n(c) = \left|c_n(x,t) - c_{n-1}(x,t)\right|$$

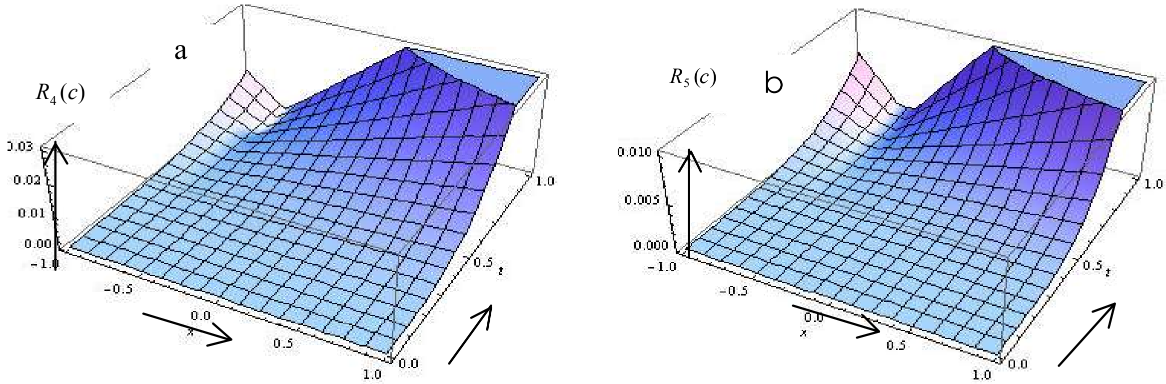

Fig. 3: Plots of residuals $R_4(c)$ and $R_5(c)$ for $\alpha = \beta = 1$

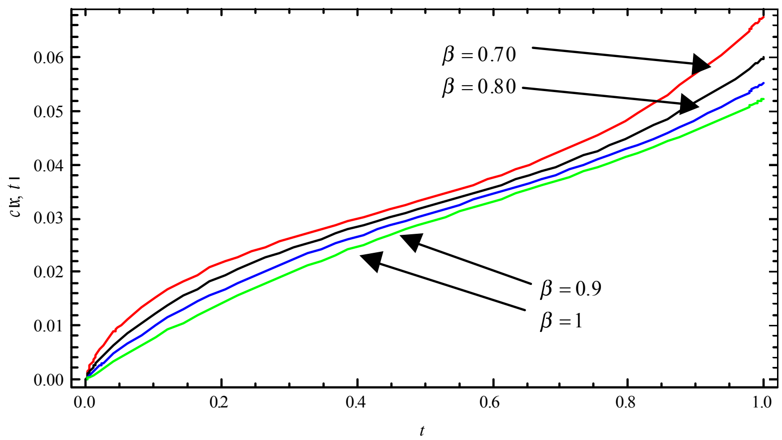

Fig. 4: Plot of c(x,t) vs. time t at x = 1 and $\alpha = 1$ for different value of β

between two consecutive terms of the approximate solutions w(x,t) and c(x,t) where $w_n(x,t)$ and $c_n(x,t)$ are the components of the approximate solutions. Residuals show that the approximate solution converges to the exact solutions. It is seen from Fig. 1(a)-(b) that residual $R_5(w)$ has more accuracy in compare of $R_4(w)$. We can find more accuracy in the approximate solutions by increasing the value of n. Figure 1(a)-(b) and 3(a)-(b) show the residual graphs for the approximate solutions w(x,t) and c(x,t) between the two consecutive terms.

The evolution results for the approximate solutions w(x,t) and c(x,t) depicted through the Fig. 2 and 4. Figure 2 show the approximate solution $w_n(x,t)$ for the different value of αat the constants value of $\beta = \alpha = 1$. It is seen that the approximate solution $w_n(x,t)$ decreases with increases in t for different value of $\alpha = 0.7, 0.8, 0.9$ and for the standard value i.e. $\alpha = 1$. Figure 4 show the approximate solution c(x,t) for the different value of β. It is seen that the approximate solution c(x,t) increases with increases in t for different value of $\beta = 0.7, 0.8, 0.9$ and for the standard value i.e. $\beta = 1$ at the constants value of $\alpha = x = 1$. It is to be noted that only fifth terms of the homotopy perturbation series were used in evaluating the approximate solutions in all figures.

Example 2: In the second example, we consider same system (4) with different initial conditions given as

$$w(x,0) = -\frac{1}{12}\left[\log\frac{(1+x)^6}{64} + x + 5\right]$$

and

$$c(x,0) = -\frac{1}{12}\left[\log\frac{(1+x)^6}{64} + 5x + 13\right] \quad (14)$$

Solving Eq. (9)-(11), with the above initial conditions, we obtain

$$w_0(x,t) = w(x,0) = -\frac{1}{12}\left[\log\frac{(1+x)^6}{64} + x + 5\right]$$

$$c_0(x,t) = c(x,0) = -\frac{1}{12}\left[\log\frac{(1+x)^6}{64} + 5x + 13\right]$$

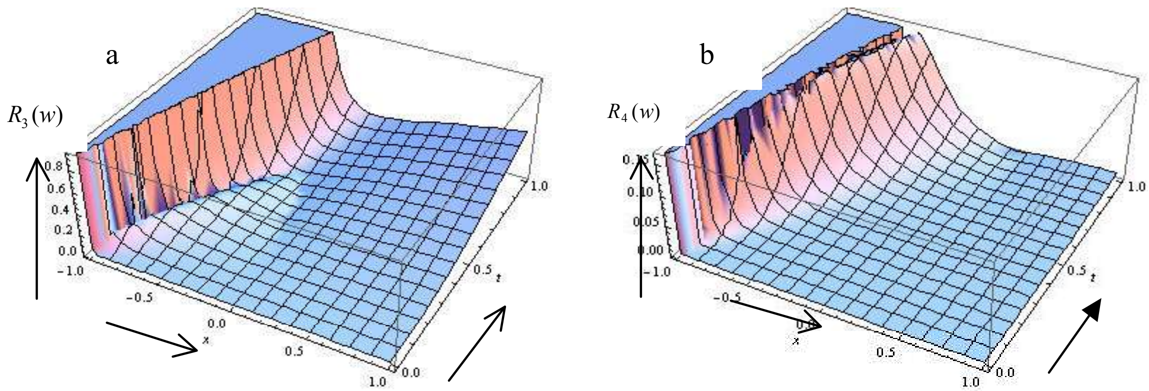

Fig. 5: Plots of residuals $R_3(w)$ and $R_4(w)$ for $\alpha = \beta = 1$

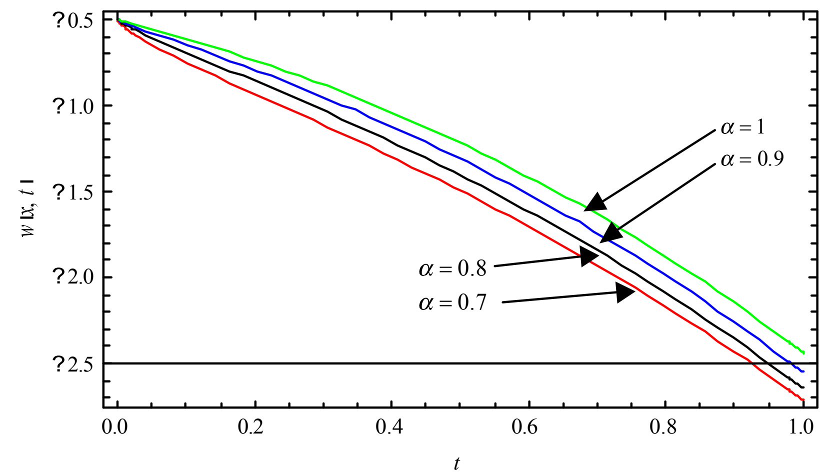

Fig. 6: Plot of w(x,t) vs. time t at x = 1 and $\beta = 1$ for different value of α

$$w_1(x,t) = -\frac{1}{144(1+x)^2}\left[\left(-72\nu + (x+1)(x+7)\left(x+5+\log\frac{(x+1)^6}{64}\right)\right) + \gamma(x+1)(5x+1)\left(5x+13+\log\frac{(x+1)^6}{64}\right)\right]\frac{t^\alpha}{\Gamma(\alpha+1)},\ldots$$

$$c_1(x,t) = \frac{1}{72(x+1)^2}\left[-5x^3 - x^2\left(47+3\log\frac{(x+1)^2}{64}\right) - x\left(115+12\log\frac{(x+1)^2}{64}\right) - 9\log\frac{(x+1)^2}{64} + 36\chi - 73\right]\frac{t^\beta}{\Gamma(\beta+1)},\ldots$$

Similarly, the rest of components of $w_n(x,t)$ and $c_n(x,t)$ can be obtained giving the series solutions. Again, we have discussed the residual

$$R_n(w) = |w_n(x,t) - w_{n-1}(x,t)|$$

and

$$R_n(c) = |c_n(x,t) - c_{n-1}(x,t)|$$

between two consecutive terms of the approximate solutions w(x,t) and c(x,t) Residuals show that the approximate solution converges to the exact solutions. It is seen fom Fig. 5(a)-(b) that residual $R_5(w)$ has more accuracy in compare of $R_4(w)$. We can find more accuracy in the approximate solutions by increasing the value of n. Figure 5(a)-(b) and 7(a)-(b) show the residual graphs for the approximate solutions w(x,t) and c(x,t) between the two consecutive terms.

Figure 6 and 8 show the approximate solutions for the different value α and of β at the constant value of x = 1. It is seen that the approximate solutions w(x,t) and c(x,t) decreases with increases in t for different value of α = 0.7, 0.8, 0.9, 1 and β = 0.7, 0.8, 0.9, 1 respectively. In this example, It is to be noted that only fifth terms of the homotopy perturbation series were used in evaluating the approximate solutions in all figures.

CONCLUSION

In this article, the homotopy perturbation method is applied to obtain the approximate solution of the

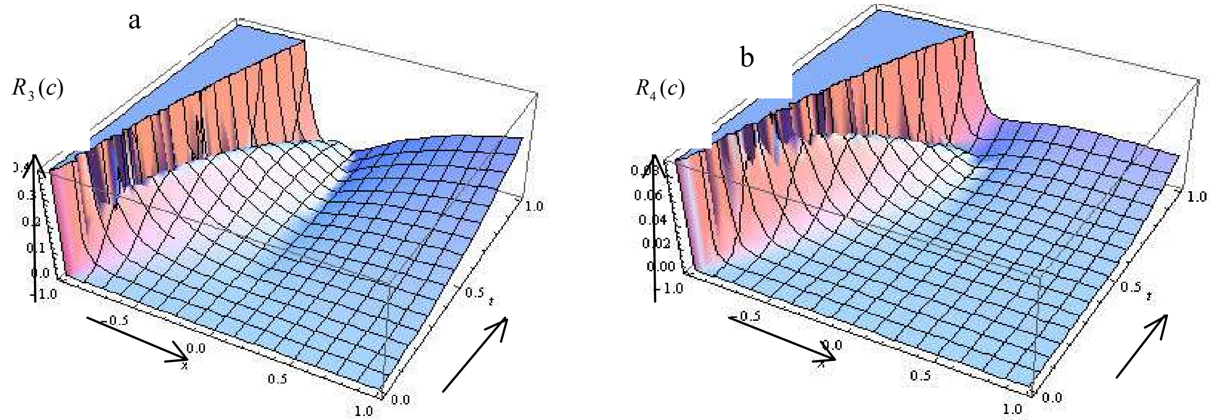

Fig. 7: Plots of residuals $R_3(c)$ and $R_4(c)$ for $\alpha = \beta = 1$

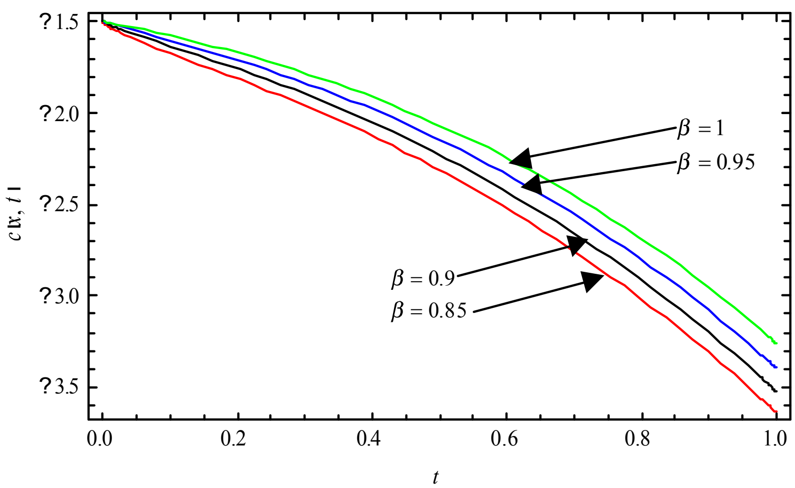

Fig. 8: Plot of $c(x,t)$ vs. time t at $x = 1$ and $\alpha = 1$ for different value of β

nonlinear system of time fractional partial differential equations of the flow rate and characteristic of the impurity. In both examples, residual graphs show that the approximate solutions converge to the exact solutions in less computational work. It is obvious that the HPM is a very powerful, easy and efficient technique for solving various kinds of nonlinear problems in science and engineering without many assumptions and restrictions. The computations associated with the example in this paper are performed using Mathematica7.

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