World Applied Sciences Journal 13 (12): 2526-2530, 2011
ISSN 1818-4952

Radial Basis Collocation Method for the Solution of Differential-difference Equations

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Abstract: In this paper, the aim is to solve the differential-difference equations using a radial basis collocation method. We present the advantages and improvement of using the proposed method for solving differential-difference equations. Some experiments are also employed to illustrate the validity and flexibility of the proposed method even where the data points are scattered.

2010 mathematics subject classification: 65L10 . 65N55

Key words: Radial basis collocation method . differential-difference equations

INTRODUCTION

Differential-Difference Equations (DDEs) arise in many areas of various mathematical modeling. For instance; infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircrafts and control problems. There are many books on the application of DDEs which we can point out to the books of Driver [1], Gopalsamy [2], Halanay [3], Kolmanovskii and Myshkis [4], Kolmanovskii and Nosov [5] and Kuang [6]. Some modelers ignore the `lag' effect and use an ODE model as a substitute for a DDE model. Kuang [6] comments under the heading ``*Small Delay Can Have Large Effects*", on the dangers that researchers risk if they ignore lags which they think are small; see also El'sgol'ts and Norkin [7]. Other modelers replace a scalar DDE by a system of ODE in an attempt to simulate phenomena more appropriately modeled by DDEs. There are inherit qualitative differences between DDEs and finite systems of ODEs that make such a strategy risky. Hence, it is bettered to discuss about the DDEs independently and try not to enter the issue of ODEs in the problem which it is a complete DDE problem. Many different methods have been presented for numerical solution of DDEs. Among these are the Radau IIA method [8], Bellman's method of steps [9], waveform relaxation method [10], Runge-Kutta method and continuous Runge-Kutta method [11, 12].

In the last two decades, the use of RBFs for both interpolation and for solving mathematical problems have received considerable attention in various fields of research and attracted many researchers to solve the problems in higher-dimensional spaces. Because the RBFs as a class of mesh-free schemes avoid grid generation and the domain of interest can be considered by a set of scattered data points among which there is no pre-defined connectivity. This method of solution is effective on scattered data points and in irregular geometries, is easy to implement in any finite dimension and is spectrally accurate.

The MQ approximation scheme is an important and an useful method using RBFs for the numerical solution of ordinary and Partial Differential Equations (ODEs and PDEs). It is a grid-free spatial approximation scheme which converges exponentially for the spatial terms of ODEs and PDEs. The MQ approximation scheme was first introduced by Hardy [13] who successfully applied this method for approximating surface and bodies from field data. Hardy [14] has written a detailed review article summarizing its explosive growth in use since it was first introduced. In 1972, Franke [15] published a detailed comparison of 29 different scattered data schemes against analytic problems. Of all the techniques tested, he concluded that MQ performed the best in accuracy, visual appeal and ease of implementation, even against various finite element schemes.

The organization of this paper is as follows. Section 2 is devoted to introduce the MQ approximation scheme and its preliminary concepts. In Section 3, we apply the MQ approximation scheme to DDEs. In Section 4, we consider the error estimation of

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the method. In Section 5, the consequences of the numerical results is shown. Finally, Section 7 consists of some obtained conclusions.

MQ APPROXIMATION SCHEME

The basic MQ approximation scheme assumes that any function can be expanded as a finite series of upper hyperboloids,

$$u(t)=\sum_{j=1}^{N} u_j \phi(t-t_j), \quad t\in R^d \tag{1}$$

where N is the total number of data centers under consideration and

$$\phi(t-t_j) = \left((t-t_j)^2+R^2\right)^{\frac{1}{2}}, \; j=1,2,\ldots,N$$

$(t\text{-}t_j)^2$ is the square of Euclidean distance in R^d and $R^2>0$ is an input shape parameter. Note that, the basis function ϕ is continuously differentiable and is a type of spline approximation.

The expansion coefficients u_j are found by solving a set of full linear equations,

$$u(t_i)=\sum_{j=1}^{N} u_j \phi(t_i-t_j), i=1,2,\ldots,N \tag{2}$$

Zerroukut *et al*. [16] found that a constant shape parameter (R^2) has achieved a better accuracy. Mai-Duy and Tran-Cong [17] have developed new methods based on Radial Basis Function Networks (RBFN) for the approximation of both functions and their first and higher derivatives. The so called direct RBFN (DRBFN) and indirect RBFN (IRBFN) methods where studied and it was found that the IRBFN methods yields consistently better results for both functions and derivatives. Recently, Aminataei and Mazarei [18] stated that, in the numerical solution of elliptic PDEs using direct and indirect RBFN methods, the IRBFN method is very accurate than other methods and the error is very small. They have shown that, especially, on one dimensional equations, IRBFN method is more accurate than DRBFN method.

Micchelli [19] proved that MQ belongs to a class of conditionally positive definite RBFN. He showed that the equation (2) is always solvable for distinct points. Madych and Nelson [20] proved that the MQ interpolation always produces a minimal semi-norm error and that the MQ interpolant and derivative estimates converge exponentially as the density of data centers increases.

In contrast, the MQ interpolant is continuously differentiable over the entire domain of data centers and the spatial derivative approximations were found to be excellent, most especially in very steep gradient regions where traditional methods fail. This excellent ability to approximate spatial derivatives is due in large part by a slight modification of the original MQ scheme by permitting the shape parameter to vary with the basis function.

Instead of using the expansion in equation (1), we have used from [21-23] as the following:

$$u(t)=\sum_{j=1}^{N} u_j \phi(t-t_j), \quad t\in R^d \tag{3}$$

where

$$\phi(t-t_j) = \left((t-t_j)^2+R_j^2\right)^{\frac{1}{2}}, j=1,2,\ldots,N \tag{4}$$

$$R_j^2 = R_{min}^2\left(\frac{R_{max}^2}{R_{min}^2}\right)^{\left(\frac{j-1}{N-1}\right)}, \; j=1,2,\ldots,N$$

and

$$R_{min}^2 > 0$$

R_{max}^2 and R_{min}^2 are two input parameters chosen so that the ratio

$$\frac{R_{max}^2}{R_{min}^2} \cong 10 \text{ to } 10^6$$

Madych [24] proved that under circumstances very large values of a shape parameter are desirable. The adhoc formula in equation (4) is a way to have at least one very large value of a shape parameter without incurring the onset of severe ill-conditioning problems.

Spatial partial derivatives of any function are formed by differentiating the spatial basis functions. Consider a one dimensional problem. The first derivative is given by simple differentiation:

$$u'(t_i) = \sum_{j=1}^{N} \frac{u_j(t_i-t_j)}{\phi_{ij}}$$

where

$$\phi_{ij} = ((t_i-t_j)^2+R_j^2)^{\frac{1}{2}}, \; i=1,2,\ldots,N$$

NUMERICAL SOLUTION OF DDES

In this section, we are interested to solve DDEs by the MQ approximation scheme mentioned in section 2. For instance, let us consider the following DDE on $[0,t_j]$,

$$u'(t)=f(t,u(t),u(t-\tau(t,u(t))),u'(t-\sigma(t,u(t)))) \qquad (5)$$

subject to

$$u(t)=\psi(t), \quad t\le 0$$

where f: $[0,R_f]\times R^3\to R$ is a smooth function, $\tau(t,u(t))$ and $\sigma(t,u(t))$ are non-negative continuous functions on $[0,t_f]\times R$ such that $0\le t-\tau(t,u(t))\le t_f$ and $0\le t-\sigma(t,u(t))\le t_f$. Also $\psi(t)$ represents the initial function or the initial data points.

For the solution of equation (5), it is sufficient to suppose that approximate solution is

$$u(t)=\sum_{j=1}^{N}u_j\phi(t-t_j), \quad 0\le t\le t_f \qquad (6)$$

Choosing t_i, i = 1,2,…,N, as collocating points, we have:

$$u(t_i)=\sum_{j=1}^{N}u_j\phi(t_i-t_j) \qquad (7)$$

$$u'(t_i)=\sum_{j=1}^{N}\frac{u_j(t_i-t_j)}{\phi_{ij}}, i=1,2,\ldots,N \qquad (8)$$

also, for i = 1,2,…,N,

$$u(t_i-\tau(t_i,u(t_i)))=\sum_{j=1}^{N}u_j\phi(t_i-\tau(t_i,\sum_{j=1}^{N}u_j\phi(t_i-t_j))-t_j) \qquad (9)$$

$$u'(t_i-\sigma(t_i,u(t_i)))=\sum_{j=1}^{N}\frac{u_j(t_i-\sigma(t_i,\sum_{j=1}^{N}u_j\phi(t_i-t_j))-t_j)}{\phi_{ij}^{(1)}} \qquad (10)$$

where

$$\phi_{ij}^{(1)}=\phi(t_i-\sigma(t_i,u(t_i))-t_j)$$
$$=((t_i-\sigma(t_i,\sum_{j=1}^{N}u_j\phi(t_i-t_j))-t_j)^2+R_j^2)^{\frac{1}{2}} \qquad (11)$$

Substituting (7), (8), (9) and (10) in (5) and imposing the supplementary condition ($u(0) = \psi(0)$) to the problem, we gain N-1 equations of differential forms and initial condition produce one equation. Hence, the system of N equations with N unknowns is available. Then we must solve this system to distinct the unknown coefficients. Hence, we have used the Gauss elimination method with total pivoting to solve such a system. Similarly, we can do the way in above for other classes of DDEs.

The best fit for approximate solution to exact solution u(t) can be studied by the role of the parameter R_j^2 in the multiquadric approximation scheme using the two parameter optimization procedure developed by Marquardt [25]. It is noteworthy that collocating points can be scattered. This is the main difference between this method of solution and other methods. In Section 2, the numerical results demonstrate this issue, easily and the efficiency of MQ approximation scheme in this sense, is observable.

Note: As shown by the numerical results in Section 2, MQ approximation scheme benefits from the following advantages:

i. Being easy to implement.
ii. Being independent of collocating points in large scales.
iii. Being well-applicable for the collocating points which have a very small metric.
iv. Requiring the minimum number of data points in the required domain.

It should be noted that the computations associated with the experiments discussed above were performed by using *Maple 13*.

NUMERICAL EXPERIMENTS

In this part, we present some experiments in-which their numerical solutions illustrate the high accuracy and efficiency of MQ approximation scheme.

Experiment 4.1: Consider the following DDE,

$$u'(t)+u(t)+u'^3(\frac{t}{3})-u(\frac{t}{2})=2t+\frac{3t^2}{4}+\frac{8t^3}{27}, \quad t\in[0,2]$$
$$u(t)=0, \quad t\le 0$$

The exact solution is $u(t) = t^2$.

For R_{max} = 1950, R_{min} = 150.26 and N = 6, we have the Table 1 which illustrate the efficiency and accuracy of MQ approximation scheme.

Table 1 shows the high agreement between exact solution and approximate solution.

In the following, we have presented an almost complicated experiment which its numerical results shows that, in spite of complexity of problem, in MQ

Table 1:

x_i	MQ approximate solution	Exact solution
0.0	0.00000000	0.00000000
0.4	0.16000000	0.16000000
0.8	0.64000000	0.64000000
1.2	1.44000000	1.44000000
1.6	2.56000000	2.56000000
2.0	4.00000000	4.00000000

Table 2:

t_i	$R_{max} = 20$ $R_{min} = 1.7$	$R_{max} = 30$ $R_{min} = 1.3$	$R_{max} = 40$ $R_{min} = 1.1$
0	1.00000002	1.00000001	1.00000001
$\frac{\pi}{10e}$	89085576	0.89085576	0.89085577
$\frac{\pi}{5e}$	0.79362401	0.79362400	0.79362400
$\frac{3\pi}{10e}$	0.70700452	0.70700453	0.70700453
$\frac{2\pi}{5e}$	0.62983906	0.62983906	0.62983906
$\frac{\pi}{2e}$	0.56109578	0.56109577	0.56109577
$\frac{3\pi}{5e}$	0.499855540	0.49985540	0.49985540
$\frac{7\pi}{10e}$	0.44529909	0.44529908	0.44529907
$\frac{4\pi}{5e}$	0.396697726	0.39669725	0.39669725
$\frac{9\pi}{10e}$	0.35340005	0.35340004	0.35340003
$\frac{\pi}{e}$	0.31482847	0.31482846	0.31482846

method, data points can be scattered. Therefore this method is not depend on the selection of points. Here, also we observe the high efficiency and accuracy of this method, too.

Experiment 4.2: Consider the following DDE,

$$
\begin{aligned}
& u'(t)+e^t u'(t-\sin(t^2))+\cos(t)u(t-\sin(t))= \\
& -e^{-t}-e^{\sin(t^2)}+\cos(t)e^{\sin(t)-t}, \quad 0\le t\le \frac{\pi}{e} \\
& u(t)=e^{-t}, \quad t\le 0
\end{aligned}
$$

without precise solution.

By choosing N = 11 and various parameters of R_{max} and R_{min}, we have the Table 2 for MQ approximate solution.

In next experiment, we show that some scattered data points are even though very closely to each other but this method of solution does not sensitive in this regard.

Table 3:

t_i	MQ approximate solutions	Exact solution
0.001	1.001000501	1.001000500
0.002	1.002002002	1.002002001
0.200	1.221402757	1.221402758
0.280	1.323129811	1.323129812
0.340	1.404947591	1.404947590
0.410	1.506817786	1.506817785
0.470	1.599994198	1.599994198
0.550	1.733253017	1.733253017
0.640	1.896480879	1.896480879
0.690	1.993715533	1.993715533
0.760	2.138276221	2.138276220
0.998	2.712850696	2.712850697
0.999	2.715564903	2.715564905
1.000	2.718281826	2.718281828

Experiment 4.3: Consider the following DDE,

$$
\begin{aligned}
& p_0 u(t)+p_1 u(\frac{t}{2})+p_2 u(t^2)+p_3 u(\sin(t))= \\
& p_0 e^t+p_1 e^{\frac{t}{2}}+p_2 e^{t^2}+p_3 e^{\sin(t)}, \quad 0\le t\le 1 \\
& u(t)=e^t, \quad t\le 0
\end{aligned}
$$

where

$$
p_0=\sqrt{e^t+\sin(t)}, p_1=\sqrt{\cos(t)}, p_2=t, p_3=1
$$

and exact solution is $u(t) = e^t$. Choosing R_{max} = 120, R_{min} = 0.499 and N = 15, the following results for scattered data are obtained.

This experiment shows that this method of solution is also well-applicable for the first three and the last three collocating points which have a very small metric.

CONCLUSIONS

In the present paper, a MQ approximation scheme is proposed to solve differential difference equations. The results reveal that the technique introduced here is effective and convenient in solving differential difference equations because this method is easy to implement and yields the desired accuracy with only a few terms. Other advantages of the present method are, a minimal number of data points in the required domain and its applicability to scattered collocated points with a very small metric. All of these advantages of the MQ approximation scheme suggest that the method is a fast and powerful tool that is more convenient for computer algorithms.

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