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Numerical Solution of Functional Differential Equations using Legendre Wavelet Method

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Abstract: In this paper, the objective is to solve the functional differential equations in the following form using Legendre Wavelet Method (LWM),

$$\begin{cases} u'(t)=f(t,u(t),u(\alpha(t))), & t_0 \le t \le t_f \\ u(t)=\phi(t), & t \le t_0 \end{cases} \tag{1}$$

where f: $[t_0, t_f]\times R^2 \to R$ is a smooth function, $\alpha(t)$ is a continuous function on $[t_0, t_f]$ and $\phi(t) \in C$ represents the initial point or the initial data. In the present paper, the most important advantages of using of the proposed method are illustrated. Some experiments are employed to illustrate the validity and flexibility of LWM, in particular for nonlinear functional differential equations.

Key words: Legendre wavelet method . functional differential equations

INTRODUCTION

Functional Differential Equations (FDEs) are considered as a branch of Delay Differential Equations (DDEs). DDEs arise in many areas of various mathematical modeling. For instance, infectious diseases, population dynamics, physiological and pharmaceutical kinetics and chemical kinetics, the navigational control of ships and aircraft and control problems. There are many books for the modeling of DDEs which we can point out to the books of Driver [1], Gopalsamy [2], Halanay [3], Kolmanovskii and Myshkis [4], Kolmanovskii and Nosov [5], Kuang [6] and El'sgol'ts and Norkin [7]. Other modelers replace a scalar DDE by a system of ODE in an attempt to simulate phenomena more appropriately modeled by DDEs.There are inherit qualitative differences between DDEs and finite systems of ODEs that make such a strategy risky. Hence, it is bettered to discuss about the DDEs independently and try not to enter the issue of ODEs in the problem which it is a complete DDE problem. Many different methods have been presented for numerical solution of DDEs such that we can point out to the Radau IIA method [8], Bellman's method of steps [9], waveform relaxation method [10], Runge-Kutta method and continuous Runge-Kutta method [11, 12].

Wavelets theory is a relatively new and an emerging area in mathematical research. It has been applied in a wide range of engineering disciplines; particularly, wavelets are very successfully used in signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation [11]. Wavelets permit the accurate representation of a variety of functions and operators. Moreover wavelets establish a connection with fast numerical algorithms [12]. In this paper, we are interested to apply the Legendre wavelet for numerical solution of Eq. (1). The properties of Legendre wavelets are utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1). The paper is organized as follows. In Section 2, we introduce the basic formulation of wavelets and Legendre wavelet. Section 3 is devoted to the solution of Eq. (1) by using Legendre wavelet. In Section 4, we present some examples which their numerical results demonstrate the accuracy and efficiency of the proposed method.

WAVELETS AND LEGENDRE WAVELET

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation

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parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [14]:

$$\psi_{a,b}(t)=|a|^{\frac{-1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R, a\neq 0$$

If we restrict the parameters a and b to discrete values as

$$a=a_0^{-k},\ b=n\,b_0 a_0^{-k}, a_0>1, b_0>0, n$$

and k are positive integers, we have the following family of discrete wavelets:

$$\psi_{k,n}(t)=|a_0|^{\frac{k}{2}}\psi(a_0^k t-nb_0)$$

where $\psi_{k,n}$ (t) form a wavelet basis for $L^2(R)$. In particular, when $a_0 = 2$ and $b_0 = 1$ then $\psi_{k,n}$ (t) forms an orthonormal basis [12].

Legendre wavelet

$$\psi_{n,m}(t)=\psi(k,\hat{n},m,t)$$

have four arguments;

$$\hat{n}=2n-1, n=1,2,3,\ldots,2^{k-1}, k$$

can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0,1) as [15]:

$$\psi_{n,m}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m(2^k t-\hat{n}), & \frac{\hat{n}-k}{2^k}\le t<\frac{\hat{n}+k}{2^k}\\ 0, & \text{else}\end{cases}$$

where,

$$m=0,1,\ldots,M-1, n=1,2,3,\ldots,2^{k-1}$$

The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b=\hat{n}2^{-k}$. Here, $P_m(t)$ are the well-known Legendre polynomials of order m which are defined on the interval [-1,1] and can be determined with the aid of the following recurrence formulae:

$$P_0(t)=1, P_1(t)$$

$$P_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)tP_m(t)-\left(\frac{m}{m+1}\right)P_{m-1}(t), m=1,2,\ldots$$

A function u(t) defined over [0,1) may be expanded as:

$$u(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\psi_{n,m}(t) \tag{2}$$

where,

$$c_{nm}=(f(t),\psi_{n,m}(t))$$

in which (.,.) denotes the inner product. If the infinite series in Eq. (2) is truncated, then it can be written as

$$u(t); \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\psi_{n,m}(t)=C^T\Psi(t) \tag{3}$$

where C and $\Psi(t)$ are $2^{k-1}M\times 1$ matrices given by

$$C=\left[c_{10},\ldots,c_{1M-1},c_{20},\ldots,c_{2M-1},\ldots,c_{2^k-1_0},\ldots,c_{2^{k-1}M-1}\right]^T$$

$$\Psi(t)=\left[\psi_{10}(t),\ldots,\psi_{1M-1}(t)\ldots,\psi_{2^{k-1}0}(t),\ldots,\psi_{2^{k-1}M-1}(t)\right]^T$$

NUMERICAL SOLUTION OF FDES

In this section, we are interested in solving the Eq. (1) i.e:

$$\begin{cases}u'(t)=f(t,u(t),u(\alpha(t))), & t_0\le t\le t_f\\ u(t)=\phi(t), & t\le t_0\end{cases} \tag{6}$$

by the Legendre wavelet method mentioned in previous section. The aforesaid equation is one of the main delay differential equations including following important pantograph equations:

$$\begin{cases}u'(t)=au(t)+bu(rt), & 0\le t\le t_f\\ u(0)=u_0\end{cases}$$

where $a,b\in C$ and $0<r<1$. This equation is a very special delay differential equation that arise in quite different fields of pure and apply mathematics such as number theory, dynamical systems, probability, quantum mechanics and electrodynamics. In particular it was used by Okendon and Taylor [13] to study how to electric current collected by the pantograph of an electric locomotive.

We first suppose that $[t_0, t_f] = [0,1]$ otherwise the problem can be mapped from $[t_0, t_f]$ to the [0,1], easily. For the solution of Eq. (2), it is sufficient to suppose that approximate solution, as mentioned in previous section, is as:

$$u(t)=C^T\Psi(t)$$

where C and $\Psi(t)$ are defined in Eqs. (4) and (5). Thus for Eq. (6), we have:

$$\begin{cases} C^T(\Psi(t))' = f(t, C^T\Psi(t), C^T\Psi(\alpha(t)) \\ C^T\Psi(0) = \phi(0) \end{cases} \quad (7)$$

Now in order to apply Legendre wavelet we are required to $2^{k-1}M$ collocation points. We have selected them in [0,1], with equal spaces

Implementing the collocation points and imposing initial value of Eq. (6) for i = 1,2,…,$2^{k-1}M-1$, we obtain:

$$\begin{cases} C^T(\Psi(t_i))' = f(t_i, C^T\Psi(t_i), C^T\Psi(\alpha(t_i)) \\ C^T\Psi(0) = \phi(0) \end{cases} \quad (8)$$

The differential equation yields $2^{k-1}M-1$ equations and initial condition produce one equation. Thus the system has $2^{k-1}M$ equations and $2^{k-1}M$ unknowns. Therefore the solution of the system the unknown coefficients is obtained.

NUMERICAL EXAMPLES

In this section, we consider three test problems corresponding to the problem given by Eq. (6) to demonstrate the efficiency of the proposed method. The computations associated with these experiments were performed in *Maple 14* on a PC with a CPU of 2.4 GHz.

Problem 4.1: Consider the following FDE,

$$\begin{cases} u'(t) - u(\frac{1}{2}t) = 0, \quad t \in [0,1] \\ u(0) = 0 \end{cases}$$

The exact solution is

$$u(t) = \sum_{k=0}^{\infty} \frac{\frac{1}{2}^{\frac{1}{2}k(k-1)}}{k!} t^k$$

We have solved this problem using LWM for $\alpha(t) = \frac{1}{2}t$. Table 1, shows the results for different M.

The results presented here,agree well with the exact solution. Also,the method yields the desired accuracy only in a few terms.

Table 1:

M-1	Maximal error
3	5.78×10^{-4}
4	7.89×10^{-6}
5	4.17×10^{-8}
6	9.42×10^{-11}
7	9.30×10^{-14}
8	2.90×10^{-15}

Table 2:

M-1	Maximal error
6	1.22×10^{-6}
7	6.30×10^{-8}
8	2.48×10^{-9}
9	9.64×10^{-11}
10	3.22×10^{-12}
11	1.26×10^{-13}
12	2.40×10^{-15}

Table 3:

M-1	Maximal error
6	1.18×10^{-5}
7	5.93×10^{-7}
8	2.43×10^{-8}
9	7.64×10^{-10}
10	4.00×10^{-11}
11	1.54×10^{-12}
12	3.43×10^{-14}

Problem 4.2: Consider the following FDE on [0,1],

$$\begin{cases} u'(t) + \sqrt{t}u(t) - e^t u(\alpha(t)) = -e^{-t} + \sqrt{t}e^{-t} - e^{t-\alpha(t)} \\ u(t) = t^2 + t + 1, \quad t \le 0 \end{cases}$$

The exact solution is u(t) = e^{-t}. We have solved this problem using LWM for different

$$\alpha(t) = \sin(\frac{t}{2})$$

and

$$\alpha(t) = t^2 + \sqrt{e^{-t}\cos(t^{10})}$$

The results are given in Tables 2 and 3, respectively.

The flexibility of the LWM can be observed again for this FDE.

Problem 4.3: Consider following nonlinear FDE on [0,1],

Table 4:

M-1	Maximal error
6	6.08×10^{-6}
8	7.48×10^{-9}
10	6.49×10^{-11}
12	3.44×10^{-14}
14	9.12×10^{-15}

Table 5:

M-1	Maximal error
6	6.11×10^{-6}
8	7.49×10^{-9}
10	6.49×10^{-11}
12	3.38×10^{-14}
14	3.50×10^{-15}

$$\begin{cases} u'(t) + e^{t^{10}} u(t) - \cos(t^{10}) u^2(\alpha(t)) = \\ e^{-t}(\cos(t) - \sin(t) + e^{t^{10}} \sin(t)) - \\ \cos(t^{10}) e^{-2\alpha(t)} \sin^2(\alpha(t)), \\ u(0) = 0 \end{cases}$$

The exact solution is $u(t) = e^{-t}\sin(t)$.

We have solved this problem using LWM for

$$\alpha(t) = \sin(\frac{t}{20}) \text{ and } \alpha(t) = t - \sin(t)$$

The results given in Table 4 and 5.

From the numerical results in Table 4 and 5, it is easy to conclude that the obtained results by the proposed method are in good agreement with the exact solution.

CONCLUSIONS

In this study, the Legendre wavelet method was proposed for solving functional differential equations. Due to orthonormality of Legendre wavelet, this method is very easy to implement and yields the desire accuracy only in a few terms. As we observed, the method works excellently for nonlinear functional differential equations, too. Therefore, we suggest using LWM for solving FDEs.

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