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Efficient Sixth-Order Nonlinear Equation Solvers Free from Derivative

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Abstract: The construction of some without memory efficient sixth-order iterative schemes for solving univariate nonlinear equations is presented. Per iteration, the novel methods comprise four evaluations of the function, while they are free from derivative calculations. The application of such iterative methods is appeared, when the cost of derivative evaluation is expensive. We analytically show the sixth-order convergence of the contributed schemes and finally numerical examples are considered to confirm their rapid convergence.

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Key words: Root solver . simple zero . derivative-free method . order of convergence . Steffensen's method

INTRODUCTION

This article deals with numerical solution of nonlinear equations of the form f(x) = 0. The exact and analytical solutions of such equations are not always at hand. That is why the accurate iterative methods, in which the number of functional evaluations is appropriate, are required. Let α be a root of such equations. This root is divided into two categories, i.e., a simple zero [6, 11, 12] or a multiple zero [3, 13]. In this study, we are concerned with simple roots of nonlinear equations. Assume α in the open interval D, be a simple root of the nonlinear equation f(x) = 0, then $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. In engineering problems or in real-world situations when the calculations of the derivatives of the functions are not a rational action or cost so much time; we require some root solvers at which there is no need of derivative calculations per iteration to obtain an accurate approximation of the exact root. Hence herein, we develop efficient sixth-order derivative-free techniques.

Let $\{x_k\}_{k=0}^{\infty}$ be a sequence in R^n, $n \geq 1$, convergent to α. Afterwards, the convergence is said to be of order p>1 (for systems of nonlinear equations; n = 1 reduces to the case of single valued nonlinear equations), if there exist M>0 and k_0 such that

$$\left\| x^{(k+1)} - \alpha \right\| \leq M \left\| x^{(k)} - \alpha \right\|^p,$$

for $k \geq k_0$. We also remind that, the Ostrowski-Traub efficiency index [16] could be provided by $p^{1/\theta}$, wherein θ is the whole number of evaluations of the iterative scheme per iteration.

This article is organized as follows. After collecting some important derivative-free root solvers in the next section, our without memory high-order algorithms will be developed in the third section. Consequently in fourth section, comparisons are made between the existed methods and the new techniques to reveal that the novel contributed derivative-free techniques are effective and convenient. Finally in the last section, our conclusion is presented.

BACKGROUND LITREATURE

For quite some time, the Steffensen's method which is given by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \qquad (1)$$

was the only reported quadratically derivative-free scheme. This method was obtained by replacing the forward-finite difference approximation in the first derivative of the well-known Newton's method.

In 2001, Wu *et al.* [17] presented another second-order derivative-free method as follows

$$x_{n+1} = x_n - \frac{f(x_n)^2}{bf(x_n)^2 + f(x_n + f(x_n)) - f(x_n)}, \qquad (2)$$

where the parameter b should be chosen such that the denominator is non-zero, for example,

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$$b = sign\{f(x_n + f(x_n)) - f(x_n)\}.$$

This scheme possesses 1.414 as the efficiency. Motivated by these methods, two-step iterative methods have been presented to date for boosting up the order of convergence and the efficiencies of the existing methods.

In 2007 in [5], a derivative-free method of order three in which we have three evaluations of the function had been presented in the following structure

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \\ x_{n+1} = x_n - \dfrac{f^3(x_n)}{[f(x_n + f(x_n))f(x_n)]\ [f(x_n) - f(y_n)]}, \end{cases} \tag{3}$$

where its efficiency is 1.442.

In 2010, another third-order iterative algorithm had been developed by Dehghan and Hajarian in [1] as follows

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)[f(y_n) + f(x_n)]}{f(x_n + f(x_n)) - f(x_n)}. \end{cases} \tag{4}$$

As we can see, this algorithm also includes three evaluations of the function per iteration with 1.442 as its efficiency.

Recently, an accurate fourth-order method [8] was proposed by Liu *et al*. as comes next

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}, \\ x_{n+1} = y_n - \dfrac{f[x_n, y_n] - f[y_n, z_n] + f[x_n, z_n]}{f[x_n, y_n]^2} f(y_n), \end{cases} \tag{5}$$

wherein $z_n = x_n + f(x_n)$ and the efficiency is 1.587. This method consists of three evaluations of the function per iteration to obtain the fourth-order convergence.

In this method $f[x_n, y_n], f[y_n, z_n], f[x_n, z_n]$ are divided differences of f(x) and defined by

$$f[x_n, y_n] = \frac{f(x_n) - f(y_n)}{x_n - y_n},$$
$$f[y_n, z_n] = \frac{f(y_n) - f(z_n)}{y_n - z_n},$$
$$f[x_n, z_n] = \frac{f(x_n) - f(z_n)}{x_n - z_n}.$$

To see more on this field of study, kindly refer to [4, 9, 10].

THE PROPOSED DERIVATIVE-FREE METHODS

In this section our contributions are derived from relation (1). Let us consider the following three-step iterative algorithm

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)^2}{f(w_n) - f(x_n)}, \\ z_n = y_n - \dfrac{f(y_n)}{f'(y_n)}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f'(z_n)}, \end{cases} \tag{6}$$

wherein $f(w_n) = f(x_n + f(x_n))$ that is to say $w_n = x_n + f(x_n)$.

This construction consists of two evaluations of the function-derivative, which is not in our aim. To remedy this, we should approximate these derivatives as efficiently as possible. Now we approximate $f'(y_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps, i.e. steps one and two. Toward this end, an estimation of the function f(t) is taken into consideration as follows

$$f(t) \approx L(t) = a_0 + a_1(t - x_n),$$

which its first derivative is $L'(t) = a_1$. By substituting in the known values

$$L(t)\,|_{x_n} = f(x_n)\,,\ L(t)\,|_{y_n} = f(y_n),$$

we could easily obtain the unknown parameters. Thus, we have $a_0 = f(x_n)$ and

$$a_1 = (f(x_n) - f(y_n))/(x_n - y_n) = f[x_n, y_n].$$

Although the use of $f(w_n)$ in approximating $f'(y_n)$ will result in a more accurate estimation, we do not use it only to reduce the computational load intentionally.

At this time, it is necessary to approximate $f'(z_n)$, in (6), with a combination of known values, that is $f(x_n)$, $f(y_n)$ and $f(z_n)$.

Note that again in our estimations we do not use the known value $f(w_n)$ by purpose. Because the usage of more known values will increases the computational complexity of the contributed method. Accordingly, we take account of an interpolating polynomial

$$f(t) \approx p(t) = b(t - x_n)^2 + c(t - x_n) + d,$$

and also consider that this approximation polynomial satisfies the interpolation conditions $f(x_n) = p(x_n)$, $f(y_n) = p(y_n)$ and $f(z_n) = p(z_n)$. By substituting the known values in p(t), we have a system of three linear equations with three unknowns. By solving this system and simplifying we have

$$\begin{cases} b = \dfrac{f[x_n, y_n] - f[x_n, z_n]}{y_n - z_n}, \\ c = \dfrac{(x_n - z_n) f[x_n, y_n] + (y_n - x_n) f[x_n, z_n]}{y_n - z_n}, \\ d = f(x_n). \end{cases} \tag{7}$$

For this reason, a powerful approximation of the first derivative of the function in the third step of (6) is attained as comes next

$$\begin{aligned} f'(z_n) &\approx p'(z_n) = 2b(z_n - x_n) + c \\ &= f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n]. \end{aligned} \tag{8}$$

By taking into consideration of the new approximation for $f'(y_n)$ and (8) in (6); and also simplifying, we attain the following three-step efficient technique, which contains only four evaluations of the function per iteration to obtain the sixth-order convergence

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[x_n, y_n]}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n]}. \end{cases} \tag{9}$$

Theorem 1: Assume that $f : D \subseteq R \to R$ be a sufficiently differentiable function with a simple zero $\alpha \subseteq D$, D be an open interval, x_0 be a guess close enough to α, then the new method (9) has sixth-order convergent.

Proof: Let $e_n = x_n - \alpha$ be the error in the nth iterate and consider $f(\alpha) = 0$, $c_k = \dfrac{f^{(k)}(\alpha)}{k!}, \forall k = 1,2,3,\ldots.$

Now we expand $f(x_n)$ around the simple zero α. Thus, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + 0(e_n^7). \tag{10}$$

By considering (10) and the first step of (9), we attain

$$x_n - \frac{f(x_n)}{f[x_n, w_n]} = \alpha + \left(-1 + \frac{1}{c_1}\right) c_2 e_n^2 + \frac{(-(2 + (-2 + c_1)c_1)c_2^2 + c_1(2 + (-3 + c_1)c_1)c_3)}{c_1^2} e_n^3 + \ldots + 0(e_n^7). \tag{11}$$

We ought to expand $f(y_n)$ around the simple root by using (11). Therefore, we have

$$f(y_n) = (1 + c_1)c_2 e_n^2 + \left(-\frac{(2 + c_1(2 + c_1))c_2^2}{c_1} + (1 + c_1)(2 + c_1)c_3\right) e_n^3 + \ldots + 0(e_n^7). \tag{12}$$

In the same way, we obtain for the second step that

$$y_n - \frac{f(y_n)}{f[x_n, y_n]} = \alpha - \frac{(1 + c_1)c_2^2}{c_1^2} e_n^3 + \frac{(-(3 + c_1(3 + c_1))c_2^3 + c_1(1 + c_1)(3 + c_1)c_2 c_3)}{c_1^3} e_n^4 + \ldots + 0(e_n^7). \tag{13}$$

This shows that the novel scheme reaches the third order of convergence at the end of the second step. At this time, the Taylor expansion about the simple root for $f(z_n)$ is needed. We find its expansion as follows

$$f(z_n) = \left(1 + \frac{1}{c_1}\right) c_2^2 e_n^3 + \frac{(-(3 + c_1(3 + c_1))c_2^3 + c_1(1 + c_1)(3 + c_1)c_2 c_3)}{c_1^2} e_n^4 + \ldots + 0(e_n^7). \tag{14}$$

Using (14) and the divided differences in the denominator of (9) gives us

$$f[x_n, z_n] + f[z_n, y_n] - f[x_n, y_n] = c_1 + \frac{(1 + c_1)c_2(2c_2^2 - c_1 c_3)}{c_1^2} e_n^3 + \ldots + 0(e_n^7). \tag{15}$$

Now dividing (14) by (15) and using the last step of (9), ends in

$$e_{n+1} = x_{n+1} - \alpha = \frac{(1+c_1)^2 c_2^3 (c_2^2 - c_1 c_3)}{c_1^5} e_n^6 + 0(e_n^7). \qquad (16)$$

This completes the proof and shows that our proposed derivative-free method is a sixth-order algorithm. (9) possesses $6^{1/4} \approx 1.565$ as its efficiency index.

Hereby we note that an iterative method without memory using four evaluations per full cycle can reach the optimal order 8 while our proposed method is a 6th scheme. Although our presented method is not optimal, it includes less computational effort than the optimal 8th order methods, such as the techniques in [2].

The attained efficiency index is greater than 1.414 of (1) and (2), 1.442 of the third-order methods, such as (3), (4) or the method in [15] and it is same to the technique in [14].

Now we aim to provide similar sixth-order schemes to (9). To do this, let us first consider a more generalized approximation of the first derivative in the Newton's iteration (at the first step) using the one parameter backward finite-difference scheme of order one in what follows:

$$f'(x_n) = \frac{f(x_n) - f(x_n - \beta f(x_n))}{\beta f(x_n)} + O(\beta f(x_n)),$$

wherein $\beta \in R - \{0\}$. Also in a similar way for approximating $f'(y_n)$ at the second step of our cycle; we get that $f'(y_n) \approx f[w_n, y_n]$ where $w_n = x_n - \beta f(x_n)$. Considering all these new approximations (and the somewhat similar estimation for the first derivative in the third step as used in (9), i.e.,

$$f'(z_n) \approx f[w_n, z_n] + f[z_n, y_n] - f[w_n, y_n],$$

we attain the follow-up sixth-order iteration without memory (uni-parameter family) in which there are only four function evaluations per full cycle

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f[x_n, w_n]}, w_n = x_n - \beta f(x_n), \\ z_n = y_n - \dfrac{f(y_n)}{f[w_n, y_n]}, \\ x_{n+1} = z_n - \dfrac{f(z_n)}{f[w_n, z_n] + f[z_n, y_n] - f[w_n, y_n]}, \end{cases} \qquad (17)$$

where its error equation is as follows:

$$e_{n+1} = \frac{c_2^3 (c_2^2 - c_1 c_3)(-1 + c_1 \beta)^4}{c_1^5} e_n^6 + 0(e_n^7), \qquad (18)$$

with $\beta \in R - \{0\}$. Thus, we have obtained another efficient sixth-order uni-parameter family of derivative-free methods, which is so much fruitful when the calculation of derivatives of the given function is expensive.

Remark 1: The without memory iterations (9) and (17) are not optimal according to the optimality conjecture of Kung and Traub (1974) [7], but they possess less computational complexity in implementing than the computational burden of the optimal eighth-order derivative-free methods.

We also remark that, iterations (9) or (17) are without memory iterative methods. To attain with memory iterative schemes, one should approximate β in (17) by iterations from the known values up to the first derivative. However, such schemes are not of our interests at the moment and thus, we do not drag the issue into them.

NUMERICAL IMPLEMENTATIONS

In this section, numerical examples are furnished to re-verify the effectiveness of the proposed derivative-free methods. The comparison among the presented method (9)-(PM1) (due to similarity of (9) and (17), we only report the numerical results of (9)), with the second-order Steffensen's method (SM), third-order method of Jain (JM), third-order method of Dehghan and Hajarian (DHM) and the fourth-order method of Liu *et al.* (LM) are given. The test functions with their roots are displayed in Table 1.

Note that in case of same-order methods, the convergence behavior is almost similar because of similar characters. The results are provided in Tables 2 and 3. All of the calculations were done with MATLAB 7.6 using 15 digits floating point arithmetic (Digits:=15). In examples considered in this article, the stopping criterion is the $|f(x_n)| \le \varepsilon$, where $\varepsilon = 10^{-15}$.

A reasonably close starting value is necessary for the methods to converge. This condition, however, practically applies to all iterative methods for solving equations.

Note that recently, many authors have employed more number of significant digits in their computations, but we herein believe that double precision is satisfactory to show the sixth-order convergence for not so close initial guesses and due to this we do not employ more number of significant digits in our numerical illustrations.

Tables 2 and 3 manifest the effectiveness of our derivative-free algorithms in this contribution. We have computed each test function by three different initial guesses. To compare the Total Number of Evaluations (TNE) of different derivative-free methods, we have provided Table 3.

Table 1: Test functions and their roots

Test functions	Roots
$f_1 = (\sin x)^2 + x$	$\alpha = 0$
$f_2 = (1+x^3)\cos(\frac{\pi x}{2}) + \sqrt{1-x^2} - \frac{2(9\sqrt{2}+7\sqrt{3})}{27}$	$\alpha = 1/3$
$f_3 = (\sin x)^2 - x^2 + 1$	$\alpha \approx 1.404491648215341$
$f_4 = e^{-x} + \sin(x) - 1$	$\alpha \approx 2.076831274533113$
$f_5 = xe^{-x} - 0.1$	$\alpha \approx 0.111832559158963$

Table 2: Comparison of different derivative-free methods in terms of required number of iterations

Methods	Guesses	SM	JM	DHM	LM	PM1
f_1	0.7	5	3	4	3	2
f_1	1.0	15	7	7	5	4
f_1	1.6	12	7	7	5	2
f_2	0.8	5	4	4	3	2
f_2	0.15	4	3	3	3	2
f_2	1.6	Div.	Div.	Div.	8	3
f_3	2.0	6	5	4	3	2
f_3	6.0	Div.	Div.	Div.	Div.	3
f_3	0.6	6	4	4	3	3
f_4	1.6	5	4	4	3	3
f_4	4.1	5	3	3	3	3
f_4	2.7	4	3	3	3	2
f_5	0.7	Div.	8	Div.	7	3
f_5	1.3	Div.	Div.	Div.	Div.	4
f_5	-1.0	14	5	9	5	3

Table 3: Comparison of TNE for different derivative-free methods

Methods	Guesses	SM	JM	DHM	LM	PM1
f_1	0.7	10	9	12	9	8
f_1	1.0	30	21	21	15	16
f_1	1.6	24	21	21	15	8
f_2	0.8	10	12	12	9	8
f_2	0.15	8	9	9	9	8
f_2	1.6	-	-	-	24	12
f_3	2.0	12	15	12	9	8
f_3	6.0	-	-	-	-	12
f_3	0.6	12	12	12	9	12
f_4	1.6	10	12	12	9	12
f_4	4.1	10	9	9	9	12
f_4	2.7	8	9	9	9	8
f_5	0.7	-	24	-	21	12
f_5	1.3	-	-	-	-	16
f_5	-1.0	28	15	27	15	12

In fact, the results from numerical experiments confirm the assertions in the last section. Scientific computations in many areas of science and engineering demand a high-order root solvers; due to this, we have constructed (9) and (17). In addition, these derivative-free methods can be further applied for finding the multiple roots of nonlinear equations by applying a suitable transformation and converting the multiple zero of the nonlinear functions to a simple root. Such derivative-free algorithms can also be extended to find the numerical solution of systems of nonlinear equations.

CONCLUDING REMARKS

The design of iterative formulas for solving nonlinear scalar equations is very important and is also an interesting task in mathematics. In other words, the analytic methods for solving such equations are almost non-existent and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iterative procedures. Hence in this contribution, some accurate and efficient without memory sixth-order derivative-free algorithms for solving single variable nonlinear equations have developed and their advantages with respect to the other existed well-known methods are illustrated by numerical examples.

The algorithms consist of four function evaluations per iteration and therefore their efficiency index is 1.565, which is greater than a lot of existing derivative-free methods'. Numerical results are in concordance with the theory developed in this paper as well. Further development to attain derivative-free methods of order 12 can be considered as future studies in this field of study by taking into account (9) and (17).

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