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# The Performance Order of Fuzzy Numbers Based on Bi-symmetrical Weighted Distance

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**Abstract:** In this paper, a new approach is prposed for ordering fuzzy numbers based on bi-symmetrical weighted distance. The proposed method considers the bi-symmetrical weighted function and the bi-symmetrical weighted distance of fuzzy numbers to rank fuzzy numbers. Some examples to compare the advantage of this approch with the existing metric index ranking methods is illustrated. The process to rank the fuzzy numbers of this method is easier than that of other efforts. This method can effectively rank various fuzzy numbers and their images and overcome the shortcomings of the previous techniques.

**Key words:** Fuzzy number . defuzzification . ranking . bi-symmetric weighted distance . strategy for decsion making

## INTRODUCTION

In many applications, ranking of fuzzy numbers is an important component of the decision process. Since Jain [19, 20] employed the concept of maximizing sets to order the fuzzy numbers in 1976, many authors have investigated various ranking methods. Some of these ranking methods have been compared and reviewed by Bortolan and Degani [2] and more recently by Chen and Hwang [14] Other contributions in this field include: An index for ordering fuzzy numbers defined by Choobineh and Li [11], ranking alternatives using fuzzy numbers studied by Dias [15] automatic ranking of fuzzy numbers using artificial neural networks proposed by Requena, *et al* [24], ranking fuzzy values with satisfaction function investigated by Lee *et al.* [10], ranking and defuzzification methods based on area compensation presented by Fortemps and Roubens [18] and ranking alternatives with fuzzy weights using maximizing and minimizing sets given by Raj and Kumar [23]. However, some of these methods are computationally complex and difficult to implement and others are counterintuitive and not discriminating. Furthermore, many of them produce different ranking outcomes for the same problem. In recent years, some researchers introduced different methods to rank fuzzy numbers based on distance between fuzzy numbers. But each method appears to have advantages as well as disadvantages. Cheng [9] proposed the distance method for ranking fuzzy numbers, that is $R(A)=\sqrt{\bar{x}^2+\bar{y}^2}$. For any two fuzzy numbers, $A_i$ and $A_j$, if $R(A_i)<R(A_j)$, then $A_i \prec A_j$; if $R(A_i) = R(A_j)$, then $A_i \sim A_j$; if $R(A_i)>R(A_j)$, then $A_i \succ A_j$. Moreover, the distance method contradicts the CV index in ranking some fuzzy numbers. Consider the three fuzzy numbers, A = (0.2,0.3,0.5), B = (0.17,0.32,0.58) and C = (0.25,0.4,0.7) utilized in Cheng [9]. In his distance method, R(A) = 0.59, R(B) = 0.60 and R(C) = 0.66, resulting in the ranking order $A \prec B \prec C$. From this result, the researchers can logically infer the ranking order of the images of these fuzzy numbers as $-A \succ -B \succ -C$. However, in the distance method, the ranking order remains $-A \prec -B \prec -C$. Obviously, the distance method also has shortcomings. Moreover, Asady in [1], developed a method based on "Sign Distance" and later a new method based on "Distance Minimization" was introduced by Asady *et al.* [1]. This method has some drawbacks, that is, for all triangular fuzzy numbers

$$u = (x_0, \sigma, \beta)$$

where

$$x_0 = \frac{\sigma - \beta}{4}$$

and also trapezoidal fuzzy numbers

$$u = (x_0, \beta_0, \sigma, \beta)$$

such that

$$x_0 + y_0 = \frac{\sigma - \beta}{2}$$

produces the same outcomes, however, it is clear that these fuzzy numbers are not ranked equally. Recently, a

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new method based on "new weighted distance", was introduced by Asady *et al.* [1]. This method, similar to Sign Distance method, considers a fuzzy origin for fuzzy numbers and then ranks them according to the distance of fuzzy numbers with respect to this origin. The above mentioned method, by defining a $\nabla$ operator, has ranked fuzzy numbers and by employing this operator, the stages of ranking has become more difficult. This article proposes a method to use the concept of bisymmetrical weighted distance without this operator to find the order of fuzzy numbers. This approach distinguishes the alternatives clearly and can be placed in the first class of Kerre's categories [30]. In this article, initialy the authors introduce a bi-symmetrical weighted distance between fuzzy numbers and secondly, suggest a new approach to the problem of defuzzification using this distance. This defuzzification can be used as a crisp approximation with respect to a fuzzy quantity. By considering this as a benchmark between fuzzy numbers, a method for ranking is presented here in. Furthermore, the ranking method suggested in this paper overcomes, to a certain extent some problems included in existing methods and possesses better efficiency of resolution and reasonability. The paper is organized as follows: In Section 2, some fundamental results on fuzzy numbers are recalled. A proposed method for ranking fuzzy numbers is in Section 3 that includes some proposed and illustrated theorems and remarks. A discussion and comparison of this effort with other methods are exhibited in Section 4. The paper concludes in Section 5 and prposes an application of this endeavor.

## PRELIMINARIES

The basic definitions of a fuzzy number are given in [21, 32] as follows:

**Definition 2.1:** Let U be a universe set. A fuzzy set $\tilde{A}$ of U is defined by a membership function $\mu_{\tilde{A}}(x) \to [0,1]$, where $\mu_{\tilde{A}}(x), \forall x \in U$, indicates the degree of x in $\tilde{A}$.

**Definition 2.2:** A fuzzy subset $\tilde{A}$ of universe set U is normal iff $\sup_{x\in U} \mu_{\tilde{A}}(x) = 1$, where U is the universe set.

**Definition 2.3:** A fuzzy set $\tilde{A}$ is a fuzzy number iff $\tilde{A}$ is normal and convex on U.

**Definition 2.4:** A trapezoidal fuzzy number $\tilde{A}$ is a fuzzy number with a membership function $\mu_{\tilde{A}}(x)$ defined by :

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} & \text{when } x \in [a_1, a_2) \\ 1 & \text{when } x \in [a_2, a_3] \\ \dfrac{a_4-x}{a_4-a_5} & \text{when } x \in (a_3, a_4] \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

which can be denoted as a quartet ($a_1$, $a_2$, $a_3$, $a_4$). In the above situations ($a_1$, $a_2$, $a_3$ and $a_4$), if $a_2 = a_3$, $\tilde{A}$ becomes a triangular fuzzy number.

**Definition 2.5:** An extended fuzzy number $\tilde{A}$ is described as any fuzzy subset of the universe set U with membership function $\mu_{\tilde{A}}$ defined as follows:

(a) $\mu_{\tilde{A}}$ is a continuous mapping from U to the closed interval [0,w], $0<w\leq 1$;
(b) $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a_1]$;
(c) $\mu_{\tilde{A}}$ is strictly increasing on $[a_1,a_2]$;
(d) $\mu_{\tilde{A}}(x) = 1$, for all $x\in[a_2,a_1]$, as w is a constant and $0<w\leq 1$;
(e) $\mu_{\tilde{A}}(x)$ is strictly decreasing on $[a_3,a_4]$ and
(f) $\mu_{\tilde{A}}(x) = 0$, for all $x \in [a_4, +\infty)$.

Furthermore, if $a_1$,$a_2$, $a_3$ and $a_4$ are real numbers and $a_1 = a_2 = a_3 = a_4$, then $\tilde{A}$ becomes a crisp real number.

**Definition 2.6:** The membership function $\mu_{\tilde{A}}$ of extended fuzzy number $\tilde{A}$ is expressed by

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu^L_{\tilde{A}}(x) & \text{when } x \in [a_1, a_2) \\ w & \text{when } x \in [a_2, a_3] \\ \mu^R_{\tilde{A}}(x) & \text{when } x \in (a_3, a_4] \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

where

$$\mu^L_{\tilde{A}}(x) : [a_1, a_2] \to [0,w]$$

and

$$\mu^R_{\tilde{A}}(x) : [a_3, a_4] \to [0,w]$$

Based on the basic theories of fuzzy numbers, $\tilde{A}$ is a normal fuzzy number if w = 1, whereas $\tilde{A}$ is a non-normal fuzzy number if $0<w\leq 1$. Therefore, the extended fuzzy number $\tilde{A}$ in definition 2.6 can be denoted as ($a_1$,$a_2$, $a_3$, $a_4$;w). The image-$\tilde{A}$ of $\tilde{A}$ can be expressed by ($-a_1$,$-a_2$,$-a_3$,$-a_4$; w) [23].

**Definition 2.7:** The r-cut of a fuzzy number $\tilde{A}$, where $0<r\leq 1$ is a set defined as

$$\tilde{A}_r = \{x \in \Re | \mu_{\tilde{A}}(x) \geq r\}$$

According to the definition of a fuzzy number it is seen once that every r-cut of a fuzzy number is a closed interval. Hence, $\tilde{A}_r = [\tilde{A}_L(r), \tilde{A}_R(r)]$, where

$$\tilde{A}_L(r) = \inf\{x \in \Re | \mu_{\tilde{A}}(x) \geq r\} \tag{2.3}$$

$$\tilde{A}_R(r) = \sup\{x \in \Re | \mu_{\tilde{A}}(x) \geq r\} \tag{2.4}$$

A space of all fuzzy numbers will be denoted by F and this article recalls that

$$\text{core}\tilde{A} = \{x \in \Re | \mu_{\tilde{A}}(x) = 1\}$$

**Definition 2.8:** A function $f$:[0,1]→[0,1] is symmetric around ½. for instance

$$f\left(\frac{1}{2}-r\right) = f\left(\frac{1}{2}+r\right)$$

for all $r \in [0, \frac{1}{2}]$, which reaches its minimum in 1/2, is called the bi-symmetrical weighted function. Moreover, the bi-symmetrical weighted function is called regular if

(1) $f\left(\frac{1}{2}\right) = 0$

(2) $f(0) = f(1) = 1$

(3) $\int_0^1 f(r)dr = \frac{1}{2}$ [25].

**Definition 2.9:** For two arbitrary fuzzy numbers $\tilde{A}$ and $\tilde{B}$ with r-cuts $[\tilde{A}_L(r), \tilde{A}_R(r)]$ and $[\tilde{\mathbf{B}}_L(r), \tilde{B}_R(r)]$, respectively, the quantity

$$d_N(\tilde{A},\tilde{B}) = \begin{pmatrix} \int_0^1 f(r)\left(\tilde{A}_L(r) - \tilde{B}_L(r)\right)^2 dr \\ +\int_0^1 f(r)\left(\tilde{A}_R(r) - \tilde{B}_R(r)\right)^2 dr \end{pmatrix}^{\frac{1}{2}} \tag{2.5}$$

where $f$:[0,1]→[0,1] is a bi-symmetrical (regular) weighted function is called the bi-symmetrical (regular) weighted distance between $\tilde{A}$ and $\tilde{B}$ based on $f$.

One can, of course, propose many regular bi-symmetrical weighted functions and hence obtain different bi-symmetrical weighted distances. Furthermore, the following function will manily be consider:

$$f(r) = \begin{cases} 1-2r \text{ when } r \in \left[0, \frac{1}{2}\right] \\ 2r-1 \text{ when } r \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{2.6}$$

## A NOVEL FUZZY ORDERING METHOD

Some researchers introduced a distance and then compared the fuzzy numbers with it, (Bardossy, *et al*, 1992; Bortlan and Degani, 2006; Cheng 1998; Tran and Duckstein, 2002; and Yao and Wu, 2000). In recent years many methods are proposed for ranking different types of fuzzy numbers (Saneifard *et al.* [28]; Saneifard 2009 [26]; Asady and Zendehnam 2007 [1]; Wang and Kerre 2001 [30]); Saneifard and Ezzati 2010 [16, 25] and can be classified into four major catagories: preference relation, fuzzy mean, spread fuzzy scoring and linguistic expression. But each method appears to have advantages as well as disadvantages. In this Section, a novel ranking fuzzy numbers method based on bi-symmetrical weighted distance is proposed and verified by some examples.

In order to rank n fuzzy numbers $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$, let the fuzzy number $\tilde{B}$ be zero in Equation 2.5 and then the parametric distance can denoted as

$$d_N(\tilde{A}_i, 0) = \left(\int_0^1 f(r)\left[\left(\tilde{A}_{i_L}(r)\right)^2 + \left(\tilde{A}_{i_R}(r)\right)^2\right]dr\right)^{\frac{1}{2}} \tag{3.1}$$

For ranking fuzzy numbers, this study defines a minimum crisp value $\tau_{min}$ to be the benchmark and its characteristic function $\mu_{\tau_{min}}(x)$ is as follows:

$$\mu_{\tau_{min}}(x) = \begin{cases} 1 \text{ when } x = \tau_{min} \\ 0 \text{ when } x \neq \tau_{min} \end{cases} \tag{3.2}$$

when ranking n fuzzy numbers $\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n$, the minimum crisp value $\tau_{min}$ is defined as:

$$\tau_{min} = \min\{x \,|\, x \in \text{Domain}(\tilde{A}_1, \tilde{A}_2, \cdots, \tilde{A}_n)\}. \tag{3.3}$$

The advantages of the definition of minimum crisp value are two-fold: first, the minimum crisp values will be obtained and also it is easy to compute. The steps of the bisymmetrical weighted distance method (BWDM) algorithm are as follows:

**Step 1:** Compute the left and right inverse functions of each fuzzy number by Equations. 2.3 and 2.4.

**Step 2:** Use Equation 3.3 to find the minimum crisp value and compute its inverse functions for the benchmark.

**Step 3:** Compute $d_N$ between fuzzy numbers and minimum crisp value ($\tau_{min}$) by Equation 2.5.

**Example 3.1:** Three fuzzy numbers $\tilde{A}$ , $\tilde{B}$ and $\tilde{C}$ have been illustrated by Chen [12] and their membership functions are shown in Table 3.1. The inverse functions calculated by Equations 2.3 and 2.4 are also shown in this table. The fuzzy numbers and the minimum crisp value are illustrated in Fig. 3.1. By Equations 2.3, 2.4 and 3.9, this study obtains $\tau_{min}$ and inverse functions

$$\tau_{min} = \min\{x \mid x \in \mathrm{Domain}(\tilde{A},\tilde{B},\tilde{C})\}$$
$$= \min\{0.2,0.4,0.5,0.6,0.7,0.8,0.9\} = 0.2$$

and

$$g_{min}(x) = \begin{cases} g^{L}_{min}(x) = 0.2 \\ g^{L}_{max}(x) = 0.2 \end{cases}$$

Utilizing Equation 2.5 the $d_N$ values between minimum crisp value and fuzzy numbers $\tilde{A}$ , $\tilde{B}$ and $\tilde{C}$ are obtained, which are equal to 0.3517, 0.3588 and 0.3657, respectively.

**Definition 3.1:** Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers characterized by definition 2.6 and $d_N(\tilde{A},\tilde{B})$ be their bi-symmetrical weighted distance.

Since the intention of the endeavor is to approximate a fuzzy number by a scalar value, the authors have used an operator $d_N{:}F\rightarrow\Re$ which transforms fuzzy numbers into a family of real line (operator $d_N$ is a crisp approximation operator). As the above defuzzification can be used as a crisp approximation of a fuzzy number, the resultant value is used to rank the fuzzy numbers. Thus, $d_N$ is used to rank fuzzy numbers therefore, the larger the $d_N$, the larger the fuzzy number.

Let $\tilde{A},\tilde{B} \in F$ be two arbitrary fuzzy numbers. The ranking of $\tilde{A}$ and $\tilde{B}$ by $d_N$ on F is defined follows:

1. $d_N(\tilde{A},\tau_{min}) > d_N(\tilde{B},\tau_{min})$ if and only if $\tilde{A} \succ \tilde{B}$
2. $d_N(\tilde{A},\tau_{min}) < d_N(\tilde{B},\tau_{min})$ if and only if $\tilde{A} \prec \tilde{B}$
3. $d_N(\tilde{A},\tau_{min}) = d_N(\tilde{B},\tau_{min})$ if and only if $\tilde{A} \sim \tilde{B}$

Table 3.1: Fuzzy numbers $\tilde{A}$ , $\tilde{B}$ and $\tilde{C}$

| Fuzzy number | membership function | inverse functions |
|---|---|---|
| $\tilde{A}$ | $\mu_{\tilde{A}}(x) = \begin{cases} 3.3x - 0.6, & 0.2 \le x \le 0.5 \\ 1, & x = 0.5 \\ 2.6 - 3.3x, & 0.5 \le x \le 0.8 \end{cases}$ | $g_{\tilde{A}}(x) = \begin{cases} \tilde{A}_L(x) = 0.3x + 0.2 \\ \tilde{A}_R(x) = 0.8 - 0.3x \end{cases}$ |
| $\tilde{B}$ | $\mu_{\tilde{B}}(x) = \begin{cases} 10x - 3, & 0.3 \le x \le 0.4 \\ 1, & x = 0.4 \\ 1.8 - 2x, & 0.4 \le x \le 0.9 \end{cases}$ | $g_{\tilde{B}}(x) = \begin{cases} \tilde{B}_L(x) = 0.1x + 0.3 \\ \tilde{B}_R(x) = 0.9 - 0.5x \end{cases}$ |
| $\tilde{C}$ | $\mu_{\tilde{B}}(x) = \begin{cases} 10x - 4, & 0.4 \le x \le 0.5 \\ 1, & 0.5 \le x \le 0.6 \\ 7 - 10x, & 0.6 \le x \le 0.7 \end{cases}$ | $g_{\tilde{C}}(x) = \begin{cases} \tilde{C}_L(x) = 0.1x + 0.4 \\ \tilde{C}_R(x) = 0.7 - 0.1x \end{cases}$ |

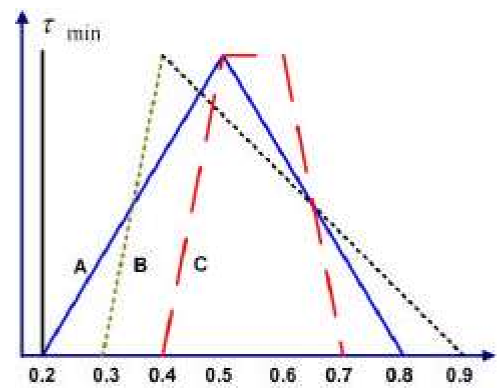


Fig. 3.1: Fuzzy numbers $\tilde{A}$ , $\tilde{B}$ , $\tilde{C}$ and $\tau_{min}$

then, this article formulates the order $\succeq$ and $\preceq$ as $\tilde{A}\succeq\tilde{B}$ if and only if $\tilde{A} \succ \tilde{B}$ or $\tilde{A} \sim \tilde{B}$ , $\tilde{A}\preceq\tilde{B}$ , if and only if $\tilde{A} \prec \tilde{B}$ or $\tilde{A} \sim \tilde{B}$ .

**Remark 3.1:** If $\mathrm{infSupp}(\tilde{A}) \ge 0$ , then $d_N(\tilde{A},\tau_{min}) \ge 0$ .

**Remark 3.2:** If $\mathrm{infSupp}(\tilde{A}) \le 0$ , then $d_N(\tilde{A},\tau_{min}) \ge 0$ .

Here, the following reasonable axioms that Wang and Kerre [30] have proposed for fuzzy quantities ranking are considered.

Let BD be an ordering method, S the set of fuzzy quantities to which the method BD can be applied and $\theta$ and $\theta'$ finite subsets of S. The statement "two elements $\tilde{A}$ and $\tilde{B}$ in $\theta$ satisfy that $\tilde{A}$ has a higher ranking than $\tilde{B}$ when BD is applied to the fuzzy quantities in $\theta$" will be written as "$\tilde{A} \succ \tilde{B}$ by BD on $\theta$", "$\tilde{A} \sim \tilde{B}$ by BD on $\theta$" and "$\tilde{A}\succeq\tilde{B}$ by BD on $\theta$" are similarly interpreted. The following axioms show the reasonable properties of the ordering approach, BD.

$A_1$. For $\tilde{A} \in \theta$ , $\tilde{A}\preceq\tilde{A}$ , by BD on $\theta$.

Table 4.1: The comparison with different ranking approaches

| Proposed method | Yager [31] | Kerre [30] | Chang [13] | Bass and Kwakernaak [3] |
|---|---|---|---|---|
| $\tilde{A}_1 \prec \tilde{A}_2$ | $\tilde{A}_1 \prec \tilde{A}_2$ | $\tilde{A}_1 \prec \tilde{A}_2$ | $\tilde{A}_1 \prec \tilde{A}_2$ | $\tilde{A}_1 \prec \tilde{A}_2$ |
| $\tilde{B}_1 \prec \tilde{B}_2$ | $\tilde{B}_1 \sim \tilde{B}_2$ | $\tilde{B}_1 \prec \tilde{B}_2$ | $\tilde{B}_1 \succ \tilde{B}_2$ | $\tilde{B}_1 \sim \tilde{B}_2$ |
| $\tilde{C}_1 \prec \tilde{C}_2 \prec \tilde{C}_3$ | $\tilde{C}_1 \prec \tilde{C}_2 \prec \tilde{C}_3$ | $\tilde{C}_1 \sim \tilde{C}_2 \prec \tilde{C}_3$ | $\tilde{C}_1 \prec \tilde{C}_2 \prec \tilde{C}_3$ | $\tilde{C}_1 \sim \tilde{C}_2 \prec \tilde{C}_3$ |
| $\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$ | $\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$ | $\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$ | $\tilde{D}_1 \prec \tilde{D}_2 \prec \tilde{D}_3$ | $\tilde{D}_1 \sim \tilde{D}_2 \prec \tilde{D}_3$ |
| $\ddot{E}_1 \succ \tilde{E}_2$ | $\ddot{E}_1 \succ \tilde{E}_2$ | $\ddot{E}_1 \succ \tilde{E}_2$ | $\ddot{E}_1 \succ \tilde{E}_2$ | $\ddot{E}_1 \prec \tilde{E}_2$ |
| $\tilde{F}_1 \succ \tilde{F}_2$ | $\tilde{F}_1 \prec \tilde{F}_2$ | $\tilde{F}_1 \prec \tilde{F}_2$ | $\tilde{F}_1 \succ \tilde{F}_2$ | $\tilde{F}_1 \prec \tilde{F}_2$ |
| $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$ | $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$ | $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$ | $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$ | $\tilde{G}_1 \prec \tilde{G}_2 \prec \tilde{G}_3$ |
| $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$ | $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$ | $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$ | $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$ | $\tilde{H}_1 \prec \tilde{H}_2 \prec \tilde{H}_3$ |
| $\tilde{I}_1 \succ \tilde{I}_2$ | $\tilde{I}_1 \sim \tilde{I}_2$ | $\tilde{I}_1 \sim \tilde{I}_2$ | $\tilde{I}_1 \succ \tilde{I}_2$ | $\tilde{I}_1 \sim \tilde{I}_2$ |
| $\tilde{J}_1 \prec \tilde{J}_2 \prec \tilde{J}_3$ | $\tilde{J}_1 \prec \tilde{J}_2 \prec \tilde{J}_3$ | $\tilde{J}_1 \prec \tilde{J}_2 \prec \tilde{J}_3$ | $\tilde{J}_1 \prec \tilde{J}_2 \prec \tilde{J}_3$ | $\tilde{J}_1 \prec \tilde{J}_2 \prec \tilde{J}_3$ |
| $\tilde{K}_1 \prec \tilde{K}_2$ | $\tilde{K}_1 \prec \tilde{K}_2$ | $\tilde{K}_1 \prec \tilde{K}_2$ | $\tilde{K}_1 \prec \tilde{K}_2$ | $\tilde{K}_1 \prec \tilde{K}_2$ |
| $\tilde{L}_1 \prec \tilde{L}_2$ | $\tilde{L}_1 \prec \tilde{L}_2$ | $\tilde{L}_1 \prec \tilde{L}_2$ | $\tilde{L}_1 \succ \tilde{L}_2$ | $\tilde{L}_1 \prec \tilde{L}_2$ |

$A_2$. For $(\tilde{A},\tilde{B}) \in \theta^2$, $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{A}$ by BD on $\theta$, should result in $\tilde{A} \sim \tilde{B}$ by BD.

$A_3$. For $(\tilde{A},\tilde{B},\tilde{C}) \in \theta^3$, $\tilde{A} \preceq \tilde{B}$ and $\tilde{B} \preceq \tilde{C}$ by BD on $\theta$, should result in $\tilde{A} \preceq \tilde{C}$ by BD.

$A_4$. For $(\tilde{A},\tilde{B} \in) \in \theta^2$, $\inf Supp(\tilde{B}) > \inf Supp(\tilde{A})$, should result in $\tilde{A} \preceq \tilde{B}$ by BD.

$A'_4$. For $(\tilde{A},\tilde{B} \in) \in \theta^2$, $\inf Supp(\tilde{B}) > \inf Supp(\tilde{A})$, should result in $\tilde{A} < \tilde{B}$ by BD on $\theta$.

**Remark 3.3:** If $\tilde{A} \preceq \tilde{B}$, then $-\tilde{A} \succeq \tilde{B}$.

Hence, this article can infer ranking order of the images of the fuzzy numbers.

## EXAMPLES

In this section, the proposed method is compared with others [5, 6, 7, 9, 22].

**Example 4.1:** First of all, this study validates its proposed method with representative examples from [3,13,31] with some advantages.

The $d_N$ values of twelve examples are shown in Fig. 4.1. Table 4.1 shows the ranking results. From this table, the main findings and BWDM with some advantages are as follows:

1. From Example L and K dipicted in Fig. 4.1, some methods use complicated and normalized processes to rank but do not obtain consistent results. However, their proposed methods are more suitable for ranking any kind of fuzzy number without a normalization process.
2. In Examples B and I (fuzzy numbers with the same mean), Yager [31], Kerre [30], Bass and Kwakernaak [3] have not been able to obtain their orderings. Chang's method [13] has been able to rank their orderings, but Chang's results violate the smaller spread and the higher ranking order. In the same Example, the researchers' proposed method can rank instantly and the results comply with human intuition.
3. In Examples C, D, and L it is obvious that the methods of Kerre [30] and Bass [3] have many limitations on triangle, trapezoid and non-normalized fuzzy numbers and so on.
4. The proposed method can be used for ranking fuzzy numbers and crisp values. But Yager has not been able to address the crisp value problem [31].
5. Kerre's method favors a fuzzy number with smaller area measurement, regardless of its relative location on the X-axis [30]. These results are not comparable with these shown in examples C and D. From Table 4.1 the proposed ranking method can correct the problem.

**Example 4.2:** The other examples in Fig. 4.2 are all positive fuzzy numbers, that can be ranked by other methods. In this case, Examples A,I, j, K and L of Tseng and Klein [29] are chosen to explain the results. They use other methods to explain the results of these methods and Table 4.2 shows the outcomes. It is evivent that most experimental results are consistent

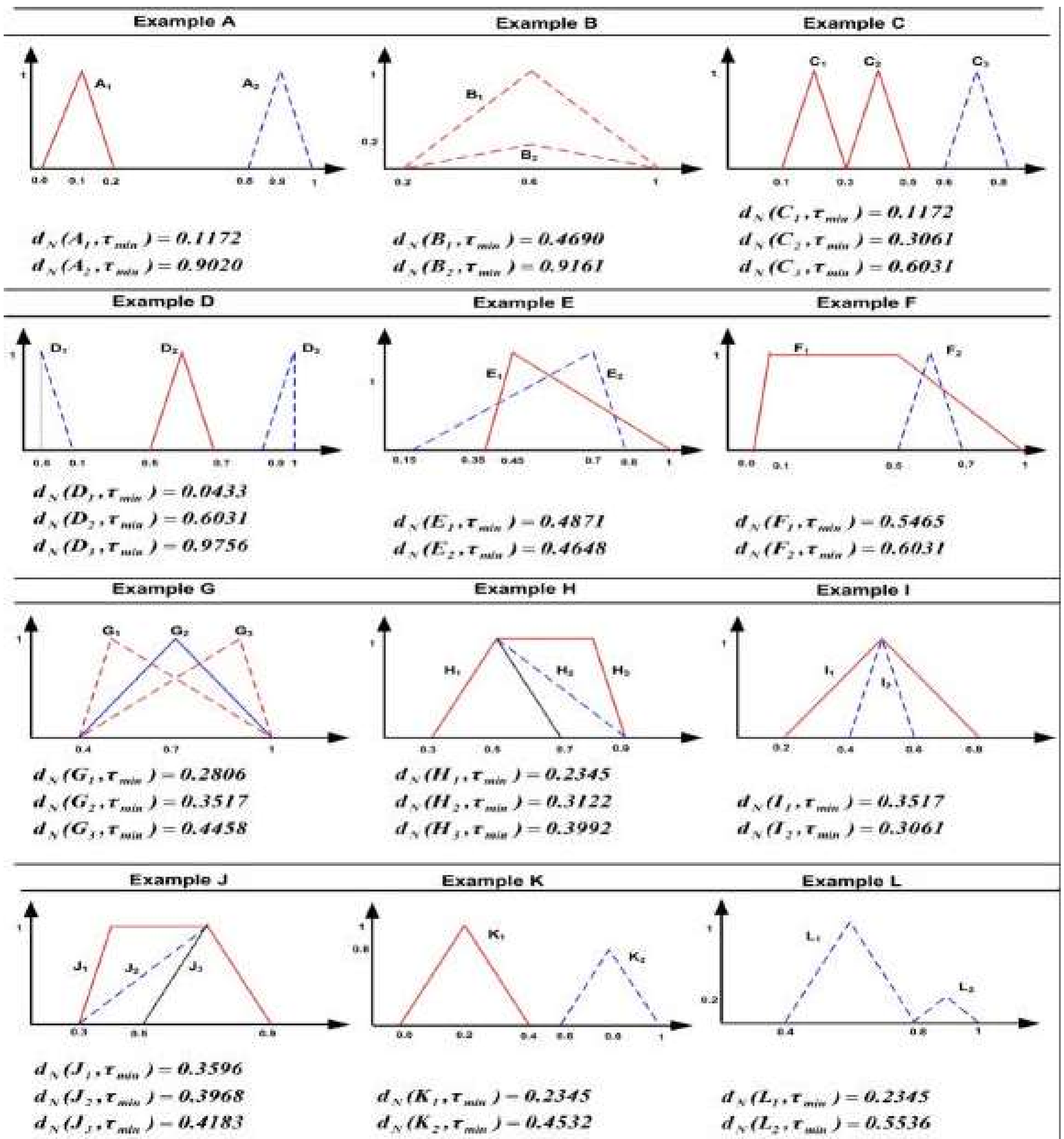


Fig. 4.1:

with other methods ( Examples B,C,D,E,H,I,j and K). Because of the outcome in Table 4.2 the results of their method are reconciled with those of other methods except for example A. In this example, Tseng and Klein [29] and Kerre [30] consider the two fuzzy numbers to be the same, but Baldwin and Guild [4] do not agree, however, both agree that $A_1$ is larger than $A_2$ and the difference is very small. Due to the different β of the Chen and Lu (2001) [8] approach, the authors may obtain the results of the ranking reversed. However, there would be a negligible difference in the outcomes of the of two methods $\tilde{j}_1$ .

**Example 4.3:** Consider the data used in Saneifard [24], that is, the three fuzzy numbers, $\tilde{A}=(5,6,6,7)$, $\tilde{B}=(5.9,6,6,7)$, $\tilde{C}=(6,6,6,7)$, as shown in Fig. 4.3. According to Equation 2.5, the ranking index values are obtained, that is, $d_N(\tilde{A},\tau_{min})=0.134$, $d_N(\tilde{B},\tau_{min})=0.155$ and $d_N(\tilde{C},\tau_{min})=0.157$. Accordingly, the ranking order of fuzzy numbers is $\tilde{C}\succ\tilde{B}\succ\tilde{A}$. However, by Chu and Tsao's approach [7], the ranking order is $\tilde{B}\succ\tilde{C}\succ\tilde{A}$. Meanwhile, using the CV index proposed by Cheng [9], the ranking order is $\tilde{A}\succ\tilde{B}\succ\tilde{C}$. From Fig. 4.3, it is easy

Table 4.2: The results of comparison using Tseng and Klein's [31] examples

| Examples | FN's | Tseng and lein | Kerre [30] | Lee and Li uniform [22] | Lee and Li proportional | Bass and Kwakernaak [3] | Chen and Lu $\beta =$ 1 | 0 | 0.50 |
|---|---|---|---|---|---|---|---|---|---|
| A | $\tilde{A}_1$ | 0.50 | 0.95 | 0.60 | 0.60 | 0.53 | -0.04 | 0.005 | 0.05 |
| | $\tilde{A}_2$ | 0.50 | 0.95 | 0.62 | 0.63 | 0.57 | 0.10 | 0.300 | 0.50 |
| B | $\tilde{B}_1$ | 0.87 | 0.99 | 0.80 | 0.80 | 0.56 | 0.10 | 0.300 | 0.50 |
| | $\tilde{B}_2$ | 0.13 | 0.54 | 0.50 | 0.50 | 0.19 | 0.30 | 0.300 | 0.30 |
| C | $\tilde{C}_1$ | 0.87 | 1.00 | 0.70 | 0.70 | 0.56 | 0.30 | 0.300 | 0.30 |
| | $\tilde{C}_2$ | 0.13 | 0.55 | 0.40 | 0.40 | 0.19 | | | |
| D | $\tilde{D}_1$ | 0.47 | 0.89 | 0.50 | 0.50 | 0.44 | 0.05 | -0.03 | -0.10 |
| | $\tilde{D}_2$ | 0.53 | 0.95 | 0.57 | 0.53 | 0.48 | | | |
| E | $\tilde{E}_1$ | 0.49 | 0.45 | 0.50 | 0.50 | 0.36 | 0.00 | 0.00 | 0.00 |
| | $\tilde{E}_2$ | 0.51 | 0.96 | 0.53 | 0.50 | 0.39 | | | |
| F | $\tilde{F}_1$ | 0.56 | 0.93 | 0.50 | 0.55 | 0.40 | 0.10 | 0.10 | 0.10 |
| | $\tilde{F}_2$ | 0.44 | 0.87 | 0.50 | 0.45 | 0.36 | | | |
| G | $\tilde{G}_1$ | 0.50 | 0.90 | 0.50 | 0.50 | 0.38 | -0.10 | 0.00 | 0.10 |
| | $\tilde{G}_2$ | 0.50 | 0.90 | 0.50 | 0.50 | 0.38 | | | |
| H | $\tilde{H}_1$ | 0.52 | 1.00 | 0.40 | 0.40 | 0.29 | 0.02 | 0.02 | 0.02 |
| | $\tilde{H}_2$ | 0.48 | 0.98 | 0.39 | 0.39 | 0.28 | | | |
| I | $\tilde{I}_1$ | 0.56 | 1.00 | 0.60 | 0.60 | 0.33 | 0.00 | 0.025 | 0.050 |
| | $\tilde{I}_2$ | 0.44 | 0.95 | 0.57 | 0.58 | 0.29 | | | |
| J | $\tilde{J}_1$ | 0.64 | 1.00 | 0.60 | 0.60 | 0.38 | 0.00 | 0.075 | 0.150 |
| | $\tilde{J}_2$ | 0.36 | 0.85 | 0.53 | 0.52 | 0.29 | | | |
| K | $\tilde{K}_1$ | 0.58 | 1.00 | 0.57 | 0.58 | 0.38 | 0.00 | 0.05 | 0.10 |
| | $\tilde{K}_2$ | 0.42 | 0.90 | 0.53 | 0.52 | 0.33 | | | |
| L | $\tilde{L}_1$ | 0.52 | 1.00 | 0.60 | 0.60 | 0.57 | 0.07 | 0.05 | 0.00 |
| | $\tilde{L}_2$ | 0.48 | 0.96 | 0.60 | 0.60 | 0.44 | | | |

to see that the ranking results obtained by the existing approaches, Cheng and Chu *et al*. [9,10] are unreasonable and are not consistent with human intuition. On the other hand, in Saneifard [25], the ranking result is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$, which is the same as the one obtained by the authors approach. However, the proposed method is simpler in the computation procedure. Based on the analysis results from Saneifard [26], the ranking results using the the new procedure and other methods are listed in Table 4.3.

**Example 4.4:** Consider the following set:

$$\tilde{A} = (1,,2,2,5)$$

$$\tilde{B} = (0,3,3,4)$$

and

$$\tilde{C} = (2,,2.5,2.5,3)$$

By using this new method, $d_N(\tilde{A}, \tau_{min}) = 0.9662$, $d_N(\tilde{B}, \tau_{min}) = 0.9969$ and $d_N(\tilde{C}, \tau_{min}) = 0.9960$. Hence, the ranking order is $\tilde{C} \prec \tilde{A} \prec \tilde{B}$. It seems that, the results obtained by the "Distance Minimization" method is unreasonable. To compare this effort with some of the other methods in Chu, *et al*, [28], the readers can refer to Table 4.4.

Furthermore, in the aforementioned example, $d_N(-\tilde{A}, \tau_{min}) = 0.44$, $d_N(-\tilde{B}, \tau_{min}) = 0.43$ and

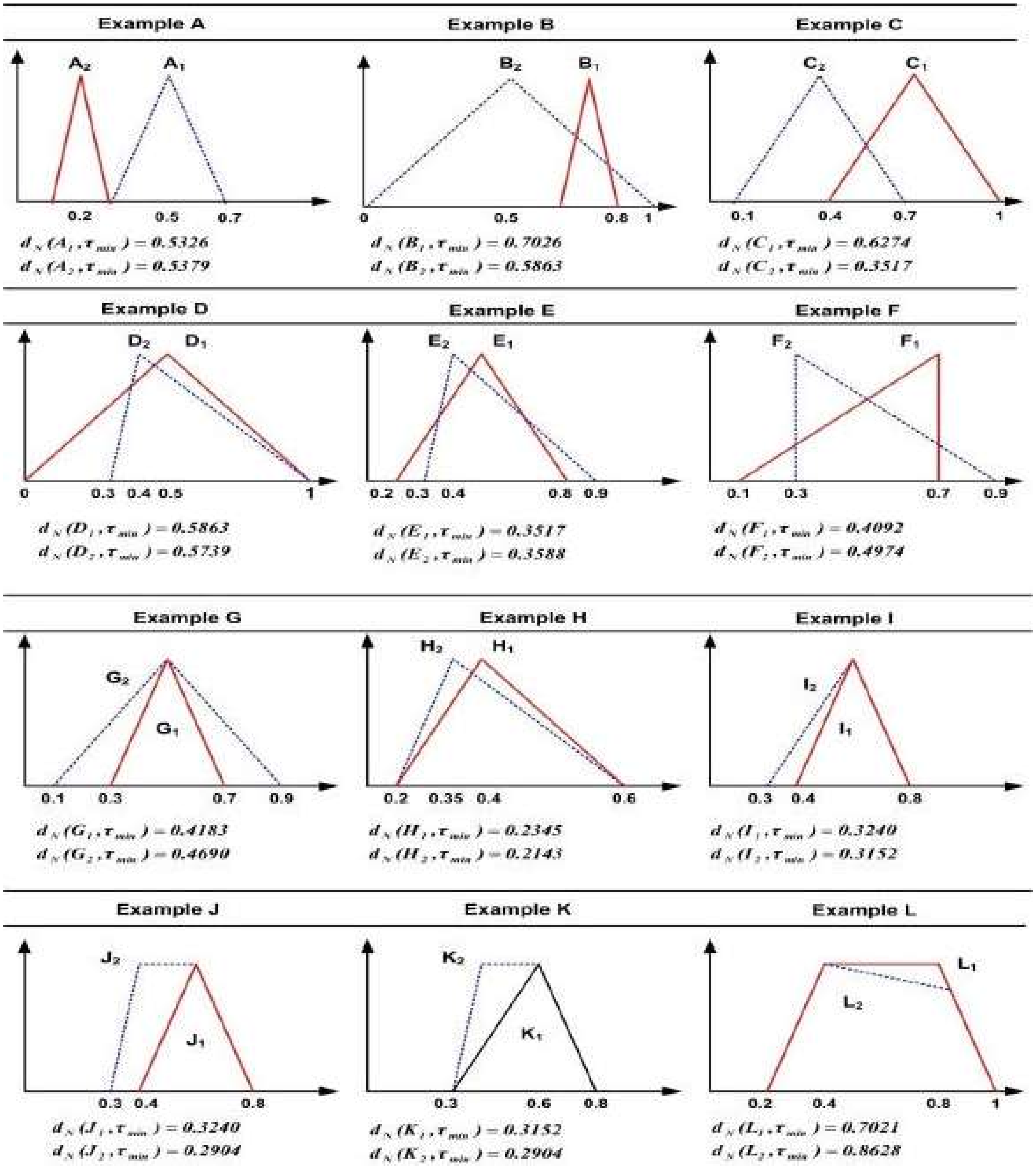


Fig. 4.2:

Table 4.3: Comparative results of example 4.3

| Fuzzy number | New approach | Sign distance with p = 1 | Sign distance with p = 2 | Chu-Tsao | Cheng distance |
|---|---|---|---|---|---|
| $\tilde{A}$ | 1.17 | 6.12 | 8.52 | 3.00 | 6.02 |
| $\tilde{B}$ | 1.28 | 12.45 | 8.82 | 3.12 | 6.34 |
| $\tilde{C}$ | 1.29 | 12.50 | 8.85 | 3.08 | 6.35 |
| Results | $\tilde{A} \prec \tilde{B} \prec \tilde{C}$ | $\tilde{A} \prec \tilde{B} \prec \tilde{C}$ | $\tilde{A} \prec \tilde{B} \prec \tilde{C}$ | $\tilde{A} \prec \tilde{C} \prec \tilde{B}$ | $\tilde{A} \prec \tilde{B} \prec \tilde{C}$ |

Table 4.4: Comparative results of example 4.4

| Fuzzy number | New approach | Sign distance with p = 2 | Distance minimization | Chu-Tsao | CV index | Magnitude method |
|---|---|---|---|---|---|---|
| $\tilde{A}$ | 0.9962 | 3.91 | 2.5 | 0.74 | 0.32 | 2.16 |
| $\tilde{B}$ | 0.9969 | 3.91 | 2.5 | 0.74 | 0.36 | 2.83 |
| $\tilde{C}$ | 0.9960 | 3.55 | 2.5 | 0.75 | 0.08 | 2.50 |
| Results | $\tilde{C} \prec \tilde{A} \prec \tilde{B}$ | $\tilde{C} \prec \tilde{A} \prec \tilde{B}$ | $\tilde{C} \prec \tilde{A} \prec \tilde{B}$ | $\tilde{A} \sim \tilde{B} \prec \tilde{C}$ | $\tilde{B} \prec \tilde{A} \prec \tilde{C}$ | $\tilde{A} \prec \tilde{C} \prec \tilde{B}$ |

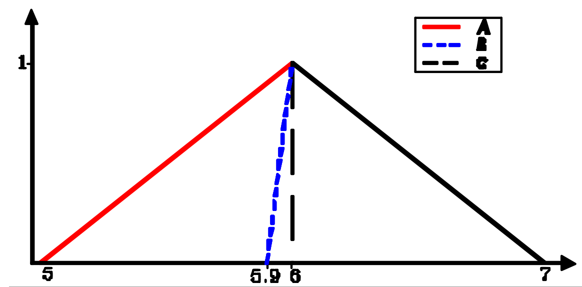


Fig. 4.3:

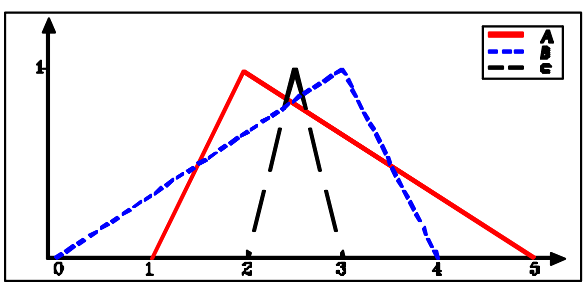


Fig. 4.4:

$d_N(-\tilde{C},\tau_{min}) = 0.5$, consequently the ranking order of the images of the three fuzzy numbers is $-\tilde{C} \succ\succ -\tilde{A} \succ -\tilde{B}$. Clearly, the proposed method has consistency in ranking fuzzy numbers and their images, which could not be guaranteed by CV-index method. With reference to Fig. 4.4, it is obvious that the new ranking method is consistent with human intuition, which contrasts the other approaches.

## CONCLUSION

In this article, the researchers propose a new defuzzification procedure using a bi-symmetrical weighted distance between two fuzzy numbers and present a modified method for ranking of fuzzy numbers. Eventhough, there is not much difference in the researchers method and others mention herein, the new approach can effectively rank various fuzzy numbers and their images. Experimental results here in reveal some advantages such as: (a) Normalizing process is unnecessary, (b) Suitable for various rankings of fuzzy numbers (without limitations) and (c) corrects Kerre's concept (regardless of its location on the X-axis). Therefore, BWDM may be applicable for practical purposes.

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