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Numerical Solution of Singular Integro-differential Equations with Cauchy Kernel

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Abstract: The main purpose of this article is to present an approximation method of for singular integro-differential equations with Cauchy kernel in the most general form under the mixed conditions in terms of the second kind Chebyshev polynomials. This method transforms mixed singular integro-differential equations with Cauchy kernel and the given conditions into matrix equation and using the zeroes of the second kind Chebyshev polynomials, the matrix equation turns a system of linear algebraic equation. The error analysis and convergence for the proposed method is also introduced. Finally, some numerical examples are presented.

Key words: The second-kind chebyshev polynomial . collocation methods . singular equation . approximation method

INTRODUCTION

The integral equations appear in many problems of physics and engineering. The singular integral equation considered to be of more interest than the other. Cauchy type singular integral equations [1, 2] was created early in the 20th century, which has undergone an intense growth during the last years. Integral equation containing singular kernel appears in studies involving airfoil [3], fracture mechanics [4] contact radiation and molecular conduction [5] and others [6-9]. Cauchy integral equations are usually difficult to solve analytically and it is required to obtain approximate solutions. So many different methods have been developed to obtain an approximate solution of a Cauchy integral equation such as iteration method [10], Berstein polynomials method [11], Jacobi polynomials method [12], Cubic spline method [13], rational functions method [14] and others. In recent years the Chebyshev polynomials have been use to find the approximate solutions of linear differential equations, linear integro-differential-difference equations and Abel equations [15-20]. In this paper, we consider the singular integro-differential equation with Cauchy kernel

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) + \frac{1}{\pi}\int_{-1}^{1} k(x,t)y(t)dt \qquad (1)$$

where

$$k(x,t) = \sqrt{\frac{1+t}{1-t}}\frac{1}{t-x}$$

with the mixed conditions

$$\sum_{k=0}^{m-1}\sum_{j=0}^{s} c^{k}_{kj} y^{(k)}(c_{kj}) = \mu_k \qquad (2)$$

and the solution is expressed in terms of the the second kind Chebyshev functions as follows:

$$y_N(x) = \sum_{r=0}^{N} a_r U_r(x), \; 0 \le i \le N \qquad (3)$$

where a_i, i=0,1,...,N are the coefficients to be determined [21]. Here $P_k(x)$ and $f(x)$ are continous functions on [-1,1], c^k_{kj} and c_{kj} are appropriate constants. In this paper, firstly, we remove the singularity, secondly the solution is expanded in terms of orthogonal polynomials. The solution of the problem reduces to the solution of a linear systems of equations. Finally, we give a numerical application to test our method.

PRELIMINARIES AND NOTATIONS

In this section, we state some basic results about polynomial approximations. These important properties will enable us to solve the singular integro differential equations. Polynomials are the only functions that the computer can evaluate exactly, so we make approximate functions R→R by polynomials. We consider real-valued functions on the compact interval [-1,1]:

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$$f:[-1,1]\to R$$

and we denote the set all real-valued polynomials on [-1,1] by P, that is

$$\forall p\in P,\ \forall x\in[-1,1], p(x)=\sum_{i=0}^{N} a_i x^i$$

and

$$P_N=\{p(x): \deg(p(x))\le N, N\in Z^+\}$$

The uniform norm (or maximum norm) is defined by

$$\|f\|_\infty = \max_{x\in[-1,1]}|f(x)|$$

Definition 2.1: For a given continuous function $f\in C[a,b]$, a best approximation polynomial of degree N is a polynomial $p_N^*(f)\in P_N$ such that

$$\|f-p_N^*(f)\|_\infty = \min\{\|f-p\|_\infty : p\in P_N\}$$

where the uniform norm is defined by $\|f\|_\infty = \max_{x\in[-1,1]}|f(x)|$.

Theorem 2.1: [20-22] Let $f\in C[a,b]$. Then for any $\varepsilon>0$, there exist a polynomial p for which

$$\|f-p\|_\infty \le \varepsilon$$

The theorem states that any continuous function f can be approximated uniformly by polynomials, no matter how badly behaved f may be on [a,b]. For phrasing; for any continuous function on [-1,1], f, there exist a sequence of polynomial $(p_N)_{N\in N}$ which converges uniformly towards f such that

$$\lim_{N\to\infty}\|f-p_N\|=0$$

Theorem 2.2: [20-22] For any $f\in[-1,1]$ and $N\ge 0$ the best approximation polynomial $p^*_N(f)$ exists and is unique.

Definition 2.2: Given a integer $N\ge 1$ a grid is a set of N+1 points $X=(x_i)_{0\le i\le N}$ in [-1,1] such that $-1\le x_0<x_1<\ldots<x_N\le 1$. Then points $(x_i)_{0\le i\le N}$ are called the nodes of the grid.

Theorem 2.3: [22-25] Given a function $f\in C[-1,1]$ and a grid of N+1 nodes $X=(x_i)_{0\le i\le N}$, there exist a unique polynomial $I_N^X(f)$ of degree N such that

$$I_N^X(f)(x_i)=f(x_i),\ 0\le i\le N$$

$I_N^X(f)$ is called the interpolant of f through the grid X.

The interpolant $I_N^X(f)$ can be express in the Lagrange form:

$$I_N^X(f)=\sum_{i=0}^{N} f(x_i)\ell_i^X(x)$$

where $\ell_i^X(x)$ is the i-th Lagrange cardinal polynomial associated with the grid X:

$$\ell_i^X(x)=\prod_{j=0,\, i\ne j}^{N}\frac{x-x_i}{x_i-x_j},\ 0\le i\le N$$

The Lagrange cardinal polynomials are such that

$$\ell_i^X(x)=\delta_{ij},\ 0\le i,j,\le N$$

The best approximation polynomials $p_N^*(f)$ is also an interpolant of f at N+1 nodes and the error in given by :

$$\|f-I_N^X(f)\|_\infty \le (1+\Lambda_N(X))\|f-p_N^*(f)\|_\infty$$

where $\Lambda_N(X)$ is the Lebesque constant relative to the grid X

$$\Lambda_N(X):=\max_{x\in[-1,1]}\sum_{i=0}^{N}|\ell_i^X(x)|$$

The Lebesque constant contains all the information on the effects of the choice of X on $\|f-I_N^X(f)\|_\infty$.

Theorem 2.4: [22, 24] For any choice of the grid X, there exist a constant C>0 such that

$$\Lambda_N(X)>\frac{2}{\pi}\ln(N+1)-C$$

Corollary 2.1: Let $\Lambda_N(X)$ be Lebesque constant relative to the grid X, then $\Lambda_N(X)\to\infty$ as $n\to\infty$.
In a similar way, by a uniform grid,

$$\Lambda_N(X)\sim\frac{2^{N+1}}{eN\ln N}\ \text{as}\ N\to\infty$$

This means that for any choice of type sampling of [-1,1], there exists a continuous function $f\in C[-1,1]$ such that $I_N^X(f)$ does not convergence uniformly towards f. Let assume that the function f is sufficiently smooth to have derivatives at least up to order N+1, with $f^{(N+1)}$ continuous i.e. $f \in C^{N+1}[a,b]$.

Definition 2.3: The nodal polynomial associated with the grid is the unique polynomial of degree (N+1) and leading coefficient 1 whose zeroes are the N+1 nodes of X:

$$w_{N+1}^{X}(x) = \prod_{i=0}^{N}(x - x_i)$$

Theorem 2.5: [22, 25] If $f \in C^{N+1}[-1,1]$, then for any grid X of N+1 nodes and for any $x \in [-1,1]$, the interpolation error is

$$f(x) - I_N^X(f)(x) = \frac{f^{(N+1)}(\varsigma)}{(N+1)!} w_{N+1}^{X}(x)$$

where $\varsigma = \varsigma(x) \in [-1,1]$ and $w_{N+1}^{X}(x)$ nodal polynomial associated with the grid X.

Definition 2.4: The Chebyshev polynomials $U_n(x)$ are the second kind polynomial in x of degree n, defined by

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \; x = \cos\theta$$

If the range of the variable x is the interval [-1,1], the range the corresponding variables θ can be taken $[0,\pi]$. These polynomials have the following properties [25]:

i) $U_n(x)$ has exactly n real zeroes on the interval [-1,1]. The m-th zero $x_{n,m}$ of $U_n(x)$ is located at

$$x_{n,m} = \cos\frac{m\pi}{n+2}$$

ii) $U_n(x)$ is orthogonal on [-1,1] with respect to the weight function $w(x) = (1-x^2)^{\frac{1}{2}}$.

iii) It is well known that the relation between the powers x^n and the second kind Chebyshev polynomials $U_n(x)$ is

$$x^n = 2^{-n}\sum_{j=0}^{\left[\frac{n}{2}\right]}\left(\binom{n}{j} - \binom{n}{j-1}\right)U_{n-2j}(x) \tag{4}$$

A particularly important class of kernels, especially in the contex of the study of Chebyshev polynomials in integral equations, comprises the Hilbert kernel

$$K(x,t) = \frac{1}{x-t} \tag{5}$$

in the neighbourhood of x = t. If [a,b] = [-1,1] then

$$K(x,t) = \frac{w(t)}{t-x} \tag{6}$$

where w(t) is one of the weight function $(1+t)^{\alpha}(1-t)^{\beta}$ with $\alpha,\beta = \pm\frac{1}{2}$.

Theorem 2.6: [25] Let $W_n(x)$ and $V_n(x)$ are the third and fourth Chebyshev polynomials respectively, then

$$W_n(x) = \frac{1}{\pi}\int_{-1}^{1} K(x,t)V_n(t)dt \tag{7}$$

where

$$K(x,t) = \frac{\sqrt{1+t}}{\sqrt{1-t}(t-x)}$$

Corollary 2.2

If $y(t) \approx \sum_{r=0}^{\infty} a_r V_r(t)$ then

$$\sum_{r=1}^{\infty} a_r W_r(x) \approx \frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1+t}}{\sqrt{1-t}(t-x)}y(t)dt \tag{8}$$

Definition 2.5: The grid $X = (x_i)_{0 \le i \le N}$ such that the x_i's are the (N+1) zeroes of the Chebyshev polynomial of degree (N+1) is called the Chebyshev-Gauss (CG) grid.

Theorem 2.7: [22, 23, 25] The polynomials of degree (N+1) and leading coefficient 1, the unique polynomial which has the smallest uniform norm on [-1,1] is the (n+1) th Chebyshev polynomial divided by 2^N.

FUNDAMENTAL RELATIONS

Let us consider Eq. (1) and find the matrix forms of the equation. First we can convert the solution $y_N(x)$ defined by a truncated second kind Chebyshev series (3) and its derivative $y_N^{(k)}(t)$ to matrix forms

$$y_N(x) = U(x)A, \; y_N^{(k)}(x) = U^{(k)}(x)A, \; k = 0,1,\cdots,N \tag{9}$$

Where,

$$U(x) = [U_0(x)\,U_1(x)\ldots U_N(x)]$$

$$A = [a_0\,a_1\ldots a_N]^T$$

By using the expression (4), taking n = 0, 1, …, N we find the corresponding matrix relation as follows

$$X^T(x) = EU^T(x) \text{ and } X(x) = U(x)E^T \tag{10}$$

where, $X(t) = [1 \ x \ldots x^N]$ and for odd N,

$$E = \begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{1}{2^2}\left(\binom{2}{1}-\binom{2}{0}\right) & 0 & \frac{1}{2^2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{2^N}\left(\binom{N}{\frac{N-1}{2}}-\binom{N}{\frac{N-3}{2}}\right) & 0 & \cdots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

for even N,

$$E = \begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{1}{2^2}\left(\binom{2}{1}-\binom{2}{0}\right) & 0 & \frac{1}{2^2}\binom{2}{0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2^N}\left(\binom{N}{\frac{N}{2}}-\binom{N}{\frac{N-2}{2}}\right) & 0 & \frac{1}{2^N}\left(\binom{N}{\frac{N-2}{2}}-\binom{N}{\frac{N-4}{2}}\right) & \cdots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

Then, by taking into account (10) we obtained

$$U(x) = X(x)(E^T)^{-1} \tag{11}$$

and

$$U^{(k)}(x) = X^{(k)}(x)(E^T)^{-1}, \ k = 0,1,\ldots,N$$

To obtain the matrix $X^{(k)}(t)$ in terms of the matrix X (t), we can use the following relation:

$$X^{(1)}(x) = X(x)B^T$$

$$X^{(2)}(x) = X^{(1)}(x)B^T = X(x)(B^T)^2$$

$$\vdots$$

$$X^{(k)}(x) = X^{(k)}(x)B^T = X(x)(B^T)^k \tag{12}$$

where

$$B = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (11) and (12) into (10) we have the matrix relation

$$y_N^{(k)}(x) = X(x)(B^T)^k(E^T)^{-1}A, \ k = 0,1,\ldots,N \tag{13}$$

Moreover, we know that [25]

$$W_n(x) = U_n(x) + U_{n-1}(x), n = 1,2,\ldots \tag{14}$$

and

$$W_0(x) = U_0(x)$$

Therefore, we get the matrix relation between W(x) and U(x)

$$W(x) = U(x)C^T$$

Where

$$C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The similar way in above procedure, the integral part (8), we obtained the matrix form,

$$\sum_{r=0}^{N} a_r W_r(x) = [W_0(x) \quad W_1(x) \quad \cdots \quad W_N(x)]A$$
$$= X(x)(E^T)^{-1}C^T A \qquad (15)$$

Matrix representation of the conditions: Using the relation (17), the matrix form of the conditions given by (2) can be written as

$$\sum_{k=0}^{m-1}\sum_{j=0}^{s} c^k{}_{kj} X(c_{kj})(B^T)^k (E^T)^{-1} A = [\mu_k] \qquad (16)$$

where

$$X(c_{kj}) = [c_{kj}^0 \quad c_{kj}^1 \quad \cdots \quad c_{kj}^N]$$

METHOD OF SOLUTION

We are ready to construct the fundamental matrix equation corresponding to Eq. (1). For this purpose, first substituting the matrix relations (13)and (15) into Eq. (1) we obtain

$$\left(\sum_{k=0}^{m} P_k(x)X(x)(B^T)^k(E^T)^{-1} - X(x)(E^T)^{-1}C^T\right)A = f(x) \qquad (17)$$

For computing the Chebyshev coefficient matrix A numerically, the zeroes of the second kind Chebyshev points defined by

$$x_{i-1} = \cos\left(\frac{i}{N+2}\right)\pi, \quad i = 1,2,\ldots,N+1$$

are putting above relation (17). We obtained

$$\left(\begin{array}{c}\sum_{k=0}^{m} P_k(x_{i-1})X(x_{i-1})(B^T)^k(E^T)^{-1} \\ -X(x_{i-1})(E^T)^{-1}C^T\end{array}\right)A = f(x_{i-1}) \qquad (18)$$

So, the fundamental matrix equation is gained

$$\left(\sum_{k=0}^{m} P_k X(B^T)^k(E^T)^{-1} - X(E^T)^{-1}C^T\right)A = F \qquad (19)$$

where

$$P_k = \begin{bmatrix} P_k(x_0) & 0 & 0 & \cdots & 0 \\ 0 & P_k(x_1) & 0 & \cdots & 0 \\ 0 & 0 & P_k(x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_k(x_N) \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}, \quad F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}$$

The fundamental matrix equation (19) for Eq.(1) corresponds to a system of (N+1) algebraic equations for the (N+1) unknown coefficients $a_0, a_1, \ldots, a_N$. Briefly, we can write Eq.(19) as

$$WA = G \text{ or } [W;G] \qquad (20)$$

so that

$$W = [w_{pq}]$$
$$= \sum_{k=0}^{m} P_k X(B^T)^k(E^T)^{-1} - X(E^T)^{-1}C^T, \; p,q = 0,1,\ldots,N \qquad (21)$$

We can obtain the matrix form for the mixed conditions (2), by means of Eq.(21), briefly, as

$$U_i A = [\lambda_i]; \text{ or } [U_i; \lambda_i] \; i = 0,1,\ldots,m-1 \qquad (22)$$

where

$$U_i = \sum_{k=0}^{m-1} c^k{}_{kj} X(c_j) B^k (M^T)^{-1} \equiv [u_{i0} \; u_{i0} \; \ldots \; u_{iN}]$$

To obtain the solution of Eq.(1) under the conditions (2), by replacing the rows matrices (22) by the last m rows of the matrix (21) we have the required augmented matrix

$$[W^*;G^*] = \begin{bmatrix} w_{00} & w_{01} & . & . & . & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & . & . & . & w_{1N} & ; & g(x_1) \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ w_{N-m,0} & w_{N-m,1} & . & . & . & w_{N-m,N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & . & . & . & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & . & . & . & u_{1N} & ; & \lambda_1 \\ \cdots & \cdots & & & & \cdots & ; & \cdots \\ u_{m-1,0} & u_{m-1,1} & . & . & . & u_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix} \qquad (23)$$

or the corresponding matrix equation

$$W^*A = G^*$$

If rank (W^*)=rank $[W^*;G^*]$=N+1, then we can write

$$A = (W^*)^{-1}G^* \; (24)$$

Thus the coefficients a_n, n=0,1,...,N are uniquely determined by Eq. (24).

Table 1: Error analysis of example 1 for the x value

x	Exact solution	N = 4	$N_e = 4$	N = 5	$N_e = 5$	N = 6	$N_e = 6$
			Present method				
-1.0	-0.33333	-0.39046	0.571E-1	-0.31275	0.205E-1	-0.32626	0.706E-2
-0.8	-0.35714	-0.38070	0.235E-1	-0.34818	0.895E-2	-0.35227	0.486E-2
-0.6	-0.38461	-0.39171	0.709E-2	-0.38083	0.377E-2	-0.38154	0.306E-2
-0.4	-0.41666	-0.41756	0.901E-3	-0.41515	0.151E-2	-0.41507	0.159E-2
-0.2	-0.45454	-0.45431	0.230E-3	-0.45412	0.415E-3	-0.45406	0.477E-3
0.0	-0.50000	-0.50000	0.100E-8	-0.50000	0.000E-0	-0.50000	0.100E-9
0.2	-0.55555	-0.55465	0.896E-3	-0.55488	0.669E-3	-0.55482	0.731E-3
0.4	-0.62500	-0.62032	0.467E-2	-0.62146	0.353E-2	-0.62128	0.371E-2
0.6	-0.71428	-0.70100	0.132E-1	-0.70365	0.106E-1	-0.70334	0.109E-1
0.8	-0.83333	-0.80272	0.306E-1	-0.80723	0.261E-1	-0.80679	0.265E-1
1.0	-1.00000	-0.93347	0.665E-1	-0.94055	0.599E-1	-0.93998	0.591E-1

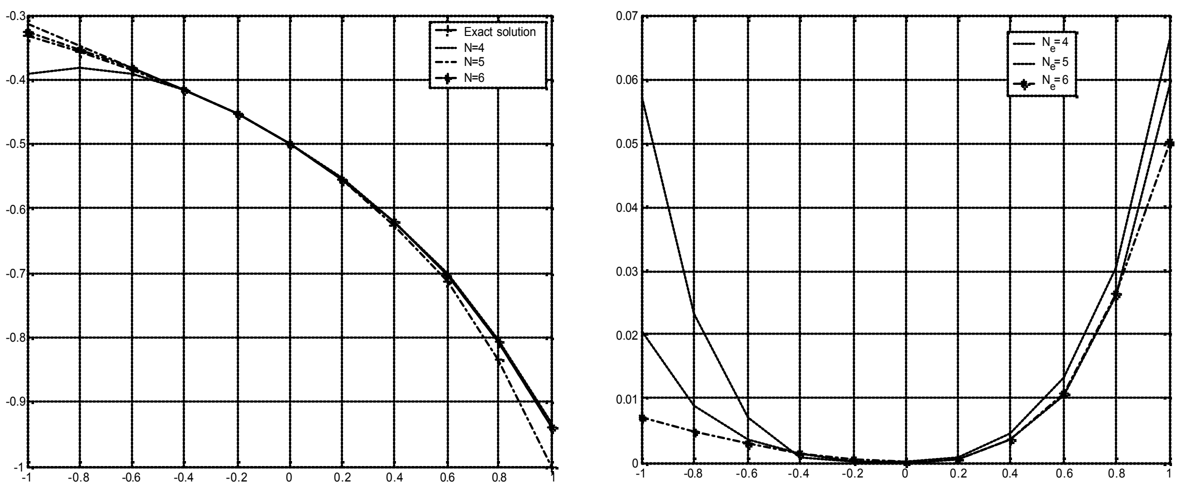

Fig. 1: Numerical solution and error function of example1 for various N

Error analysis and convergence: Since, $\|U_{N+1}\|_\infty = N+2$, we conclude that if we choose the grid nodes $(x_i)_{0\le i\le N}$ to be zero the (N+1) zeroes of the Chebyshev polynomials U_{N+1}, we have

$$\left\| w^{X}_{N+1} \right\| = \frac{N+2}{2^N}$$

and this is the smallest possible value. In particular, from Theorem 2.10, for any $y \in C^{N+1}[-1,1]$ we have

$$\|y - y_N\|_\infty \le \frac{N+2}{2^N(N+1)!}\left\|f^{N+1}\right\|_\infty$$

If $y^{(N+1)}$ is uniformly bounded, the convergence of the interpolation y_N towards y when $N\to\infty$ is then extremly fast.

ILLUSTRATIVE EXAMPLE

In this section, several numerical examples are given to illustrate the accuracy and effectiveness properties of the method and all of them were performed on the computer using a program written in Maple 9. The absolute errors in tables are the values of $N_e = |y(x)-y_N(x)|$ at selected points.

Example 5.1: Consider the following singuler integro-differential equation with Cauchy kernel

$$(x+2)^3 y'' + 2(x-2)y' + y = \frac{\sqrt{3}-1}{x-2} - \frac{1}{\pi}\int_{-1}^{1} k(x,t)y(t)dt$$

with $y(0) = -\frac{1}{2}$, $y'(0) = -\frac{1}{4}$

and its exact solution is $y(x)=\frac{1}{x-2}$. We obtained the approximate solution of the problem for N = 8, 9, 10 which are tabulated and graphed. For numerical results Table 1. We display a plot of absolute difference exact and approximate solutions with error functions for various N are shown in Fig. 1.

Example 5.2: Consider the following singuler integro-differential equation with Cauchy kernel

$$xy''-(x-1)y'+y=8x^2-4x-\frac{3}{2}+\frac{1}{\pi}\int_{-1}^{1}k(x,t)y(t)dt$$

with y(0) = 1, y′(0) = -1 and this equation has the exact solution $y(x) = x^2-x+1$. For N = 5, we obtained exact solution.

CONCLUSION

A new method based on the truncated Chebyshev series of the second kind is developed to numerical solve singular integro-differential equations with mixed conditions on Chebyshev-Gauss grid. Singular integro-differential equations and singular integral equations are usually difficult to solve analytically. In many cases, it is required to obtained the approximate solution. For this propose, the present method can be proposed.In this paper, the second kind Chebyshev polynomial approach has been used for the approximate solution of singular integro-differential equations. For the suggested method, we show error analysis and converge. Thus the proposed method is suggested as an efficient. Examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method very attractive and contributed to the good agreement between approximate and exact values in the numerical example.

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