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On Finding Solutions of Fredholm Integral Equations by Branching Process Simulation

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Abstract: In this paper, we introduce a stochastic method for solving nonlinear Fredholm integral equations. This kind of integral equations sometimes do not have straightforward and even unique solution. We employ the branching process to solve such integral equations. Finally, a numerical example is provided to show that the proposed is effective.

Key words: Branching process · Monte Carlo method · Fredholm integral equations · Markov chain · Simulation
AMS Classification: 34L16 · 60J65 · 60Gxx · 37M05

INTRODUCTION

There are several approach to solve Fredholm integral equations, however they have usually results acceptable solutions in low dimensions. In probability theory, a branching stochastic process is a Markov process that models a population in which each individual in generation n produces some random number of individuals in $n + 1$, according to a fixed probability distribution that does not vary from individual to individual [1, 2]. Monte Carlo simulation is a classical tool for solving high dimensional problems and has been successfully used in different areas, for example see [3, 4].

We consider the following function

$$j(u) \equiv (g,u) = \int_G g(x)u(x)dx$$

Where the domain $G \subseteq R^n$ and the point $x = (x_1, x_2, ..., x_n) \in G$ is a point in Euclidean space and g(x), u(x) belong to Banach space, mark that g(x) has been supposed Dirac delta due to the special usage of this function is sampling from probability density function.

$$u(x) = f(x) + \lambda \iint_G ... \int k(x, y_1, y_2 ..., y_m) \prod_{i=1}^{m} u(x_i) \prod_{i=1}^{m} dx_i.$$

The general model of Fredholm integral equation is as follow

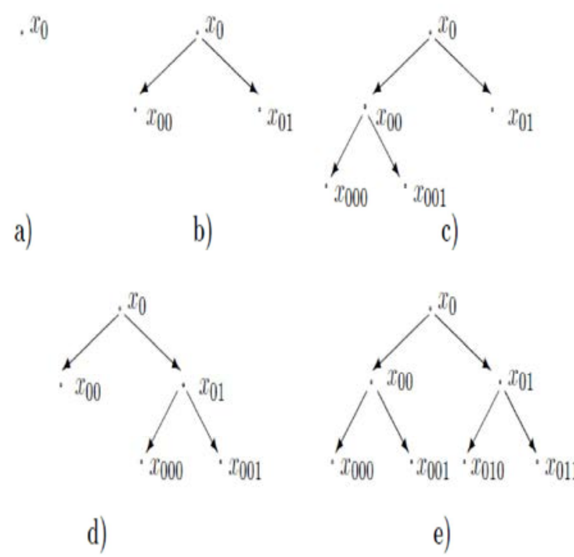

Fig. 1: The first two iterations of random tree

Now, we consider this model for double integral equation of the second kind

$$u(x) = f(x) + \lambda \iint_G k(x, y, z)u(y)u(z)dydz.$$

The equation (3) converges if

$$\left\| k(u^m) \right\| \le \max \iint_G \dots \int \left| k(x, y_1, ..., y_m) \right| \prod_{i=1}^{m} dy_i \le \frac{1}{m}.$$

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Monte Carlo Method for Solving Nonlinear Fredholm Integral Equation: In this section we explain a relation between branching process and solving nonlinear integral equations. Consider the model of branching process that just has been divided in two branches. It begins with point x_0, then x_0 generates the next generations x_{00} and x_{01}. All generated particles at the next moment behave as the initial one. (Fig. 1)

Each particle begins generating with probability $p_m(x_0)$ and die out with probability $h(x_0)$such that [2].

$$p_m(x_0) = 1 - h(x_0)$$

This probability is initial probability of each step and it has direct relation with dying out probability. Moreover, for transition probability we have

- $p_m(x) \geq 0$
- $p(x_0, x_{00}, \ldots, x_{0m-1}) \geq 0$
- $$\int \ldots \int p(x_0, x_{00}, \ldots, x_{0m-1}) \prod_{i=0}^{m} dx_{0i} = 1.$$

Associated with the sample path $x_0 \to x_1 \to x_2 \to \ldots \to x_n$,

Where n is a given integer number, the following random variable defined by

$$\Gamma_n(h) = \frac{h(x_0)}{p(x_0)} \sum_{m=0}^{n} w_m f(x_m)$$

Where

$$w_m = w_{m-1} \frac{k(x_{m-1}, x_m)}{p(x_{m-1}, x_m)}$$

Using the model in Fig. 1, we consider the following relations for solving Fredholm integral equations. We have

$$u_0(x_0) = f(x_0)$$
$$u_1(x_0) = f(x_0) + \iint k(x_0, x_{00}, x_{01}) f(x_{00}) f(x_{01}) dx_{00} dx_{01}$$
$$u_2(x_0) = f(x_0) + \iint k(x_0, x_{00}, x_{01}) f(x_{00}) f(x_{01}) dx_{00} dx_{01} +$$
$$\iint k(x_0, x_{00}, x_{01}) f(x_{01}) \times \iint k(x_{00}, x_{000}, x_{001}) f(x_{000}) f(x_{001}) dx_{00} dx_{01} +$$
$$\iint k(x_0, x_{00}, x_{01}) f(x_{01}) \iint k(x_{01}, x_{010}, x_{011}) f(x_{010}) f(x_{011}) dx_{010} dx_{011}$$
$$\times \iint k(x_{00}, x_{000}, x_{001}) f(x_{000}) f(x_{001}) dx_{00} dx_{01}.$$

Definition 2.1 Full random tree with L generations is called the tree Γ_L where the dying out of particles is not visible from zero to $(L-1)^{\text{th}}$ generation, but all the generated particles of the L^{th} generation dies out.

If the process has been stopped at the initial point, then $u_0(x_0) = f(x_0)$. Therefore the Monte Carlo estimation is

$$\theta_g(\gamma_0) = \frac{g(x_0) f(x_0)}{p_0(x_0) h(x_0)}.$$

The full model of Monte Carlo estimator is

$$\theta_g(\gamma_0) = \frac{g(x_0)}{p_0(x_0)} \times \frac{k(x_0, x_{00}, x_{01})}{p_2(x_0) p(x_0, x_{00}, x_{01})} \times \frac{k(x_{00}, x_{000}, x_{001})}{p_2(x_{00}) p(x_{00}, x_{000}, x_{001})}$$
$$\times \frac{k(x_{01}, x_{010}, x_{011})}{p_2(x_{01}) p(x_{01}, x_{010}, x_{011})} \times \frac{f(x_{000}) f(x_{001}) f(x_{010}) f(x_{011})}{h(x_{000}) h(x_{000}) h(x_{000}) h(x_{000})}$$

Theorem 2.1: The mathematical expectation of the random variable $\theta_g(\gamma_0)$ is equal to function $J(u_L)$ i.e.,

$$E\theta_g\left(\frac{\gamma}{\Gamma_L}\right) = J(u_L) \equiv (g, u_L).$$

Proof: See [5].

Lemma 2.1: The transition density function

$$p(x, y_1, \ldots, y_m) = \frac{\left| k(x, y_1, \ldots, y_m) \prod_{i=1}^{m} u(y_i) \right|}{\int \ldots \int \left| k(x, y_1, \ldots, y_m) \prod_{i=1}^{m} u(y_i) \right| \prod_{i=1}^{m} dy_i}$$

minimizes u(x) for any $x \in G$, where u(x) is solution of (4).

Proof: See [5].

Lemma 2.2: The initial frequency function

$$p_0(x_0) = \frac{|g(x_0)\phi(x_0)|}{\int g(x_0)\phi(x_0) dx_0}$$

minimizes the functional $\int g^2(x_0) u^2(x_0) p^{-1}(x_0)$.

The minimum of this functional is equal to

$$\left(\int |g(x_0)u(x_0)|dx_0\right)^2$$

Proof: See [6].

Theorem 2.3: If $l \to \infty$, then we have

$$\lim_{l\to\infty} E\theta_g\left(\frac{\gamma}{\Gamma_l}\right) = \int_G g(x)u(x)dx$$

Proof: See [5].

In the rest of this paper, for simplicity, suppose that the kernel of Fredholm integral equation (3) is separable. Also, we have supposed that all random trees have a finite number of generations and the average value of the particles which are born in any generation is finite.

Algorithm:

- Generate random variable $\xi, \xi_1, \xi_2, ..., \xi_k$ from uniform distribution.
- Calculate $\phi(x_0) = \frac{|f(x_0)|}{\sqrt{h(x_0)}}$.
- Compute initial distribution

$$p_0(x_0) = \frac{|g(x_0)\phi(x_0)|}{\int g(x_0)\phi(x_0)dx_0}.$$

- Compute die out probability $p_m(x) = 1 - h(x_0)$.
- If the die out probability at initial point multiplied by $\frac{f(x_0)}{h(x_0)}$ which is die out probability

$$\theta_g(\gamma) = \frac{g(x_0)f(x_0)}{p_0(x_0)h(x_0)}$$

- Calculate optimal transition density function

$$p(x_\xi, x_{\xi 1}, x_{\xi 2}) = \frac{k(x_\varepsilon, x_{\xi 1}, x_{\xi 2})}{\iint_G k(x_\xi, x_{\xi 1}, x_{\xi 2})dx_{\xi 1}dx_{\xi 2}}$$

- Calculate

$$\theta_g(\gamma_0) = \frac{g(x_0)}{p_0(x_0)} \times \frac{k(x_0, x_{00}, x_{01})}{p_2(x_0)p(x_0, x_{00}, x_{01})} \times \frac{k(x_{00}, x_{000}, x_{001})}{p_2(x_{00})p(x_{00}, x_{000}, x_{001})} \times \frac{k(x_{01}, x_{010}, x_{011})}{p_2(x_{01})p(x_{01}, x_{010}, x_{011})} \times \frac{f(x_{000})f(x_{001})f(x_{010})f(x_{011})}{h(x_{000})h(x_{000})h(x_{000})h(x_{000})}$$

- If $|\theta_g(\gamma) - u^*(x)| \leq \delta$ then $\theta_g(\gamma)$ is the estimator of Monte Carlo else go to step 1.
- Stop the algorithm when all points die out.

Numerical Experiments: In this section, show the performance of the algorithm to obtain the unique solution of the following integral equations. We run our results on workstation, Intel(R) 1.83 GHz Dual CPU, 2.00 GB RAM using MATLAB software.

Table 1: Monte Carlo solution and relative error for three transition density functions using N=5000

Transition density functions	$p_2(x) = 0.25$		$p_2(x) = 0.5$		$p_2(x) = x/4$	
x	MC solution	Relative error	MC solution	Relative error	MC solution	Relative error
0.0	0.9917	0.0083	0.9980	0.0020	1.00	0.00
0.1	1.0063	0.0063	1.0192	0.0192	0.9999	1.459e-4
0.2	0.9893	0.0107	1.0090	0.0090	1.0007	7.417e-4
0.3	1.0022	0.0022	1.0041	0.0041	1.0015	0.0015
0.4	1.0021	0.0021	0.9982	0.0018	1.0003	2.647e-4
0.5	1.0125	0.0125	1.0412	0.0412	0.9998	1.869e-4
0.6	1.0067	0.0067	1.0260	0.0260	0.9999	1.314e-4
0.7	1.0088	0.0088	1.0334	0.0334	1.0004	3.550e-4
0.8	1.0125	0.0125	1.0116	0.0116	1.0006	5.672e-4
0.9	1.0084	0.0084	1.0337	0.0337	1.0042	0.0042
1.0	1.0002	1.92e-4	1.0367	0.0367	0.9996	3.741e-4

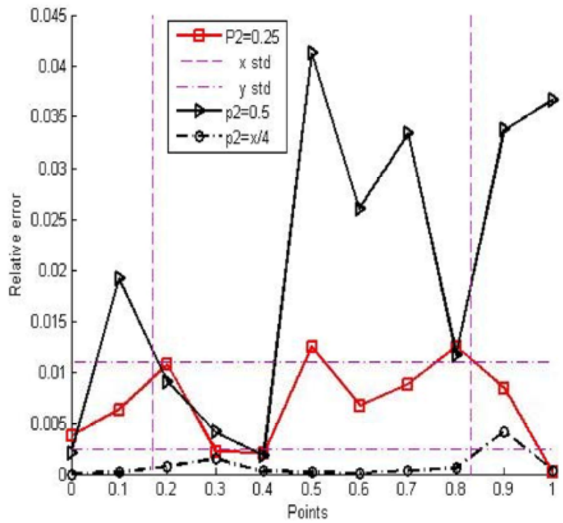

Fig. 2: Comparison of relative error for tree different ransition density function.

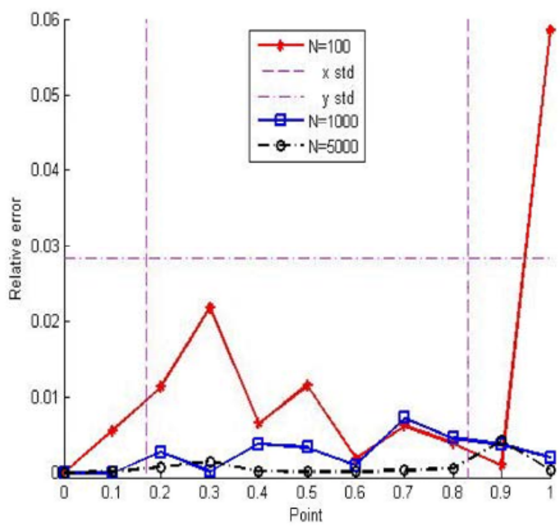

Fig. 3: Comparison of relative error for different random trees.

We can see in Fig. 1 with the increasing number of random trees the Monte Carlo solution converges asymptotically to the exact one. Also, the experimental results for three transition density (0.25, 0.5, x/4), are outlined in Table 2.

Example: Consider the following integral equation,

$$u(x) = 1 - 0.1667x + 0.0094\iint_D x(8y - z)u(y)u(z)dydz$$

Where $D = [0,1]\times[0,1]$, $x \in \square$ and the exact solution is $u(\mathrm{x}) = 1$.

We have simulated tree in Fig. 1 100, 1000, 5000 with three transition density functions $p_2(x) = 0.25$, $p_2(x) = 0.5$, and $p_2(x) = x/4$. The best results are for 5000 and $p_2(x) = x/4$ as summarized in Table 1 and Fig 2.

CONCLUSIONS

As it knows, there are no efficient numerical algorithms to solve seemed kind Fredholm integral equations. In this paper, we have proposed a branching process based algorithm to solve Fredholm integral equations of the second kind. Finally, several numerical examples are presented.

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