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HAMAD Formulations: General Formulations for Exact and Similarity Transformations of ODEs and PDEs

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Abstract: The current paper outlines a new algorithm, which is used to determine the exact solutions of ODEs and the similarity transformations of PDEs whose the invariant groups are known. Three groups of formulations (named by HAMAD Formulations) are presented: (I) Formulations for the exact solutions of ODE. (II) Formulations for similarity transformations of PDEs contains two independent variables. (III) Formulations for similarity transformations of PDEs contains three independent variables. By using these formulations enable readers can calculate the similarity transformations which transform PDEs to ODEs and also can calculate exact solutions of ODEs. Some examples are presented.

Key words: Group theory . HAMAD formulations

INTRODUCTION

Group theoretic methods provide a powerful tool because they are not based on linear operators, superposition, or any other aspects of linear solution techniques. Therefore, these methods are applicable to nonlinear differential models. The concept of a symmetry of a differential equation was introduced by Sophus Lie at the end of the 19th century while he was searching for a general theory of solving differential equations Schwarz [1]. Although Lie has obtained numerous results for partial differential equations, only the concept of a similarity transformation, introduced as a systematic method about 50 years after Lie's death by Birkhoff [2], caused broader applications and led to a better understanding of various apparently unrelated results.

Mathematicians, Engineers and Scientists often try to find ways and means to reduce the partial differential equations into ordinary differential equations in order to simplify the solution technique. The governing partial differential equations are often solved using similarity methods. Using similarity methods, the number of independent variables are reduced and the equations are transformed into ordinary differential equations which can be more easily solved.

Group analysis is the only rigorous mathematical method to find all symmetries of a given differential equation and no ad hoc assumptions or prior knowledge of the equation under investigation is needed. Morgan [3] presented a theory which led to improvements over earlier similarity methods. Birkhoff [4, 5] was the first author introduced group methods as a class of methods leads to a reduction of the number of independent variables. Moran and Gaggioli [6, 7] presented general systematic group formalism for similarity analysis, where a governing system of partial differential equations was reduced to a system of ordinary differential equations.

This paper presents a simple method for obtaining the exact solution of ODEs and the similarity transformations which transform PDEs to ODEs.

MATHEMATICAL SIMULATION

The Mathematical Algorithm: Here we present the main steps which apply to ODEs or PDEs to find the similarity forms.

A diffeomorphism

$$\Gamma :(x_j,\, y_k) \mapsto (\bar{x}_j, \bar{y}_k), \qquad j = 1,2,\ldots,s, k = 1,2,\ldots,m$$

is a symmetry of the PDE

$$f(x_j, y_k, \frac{\partial y_k}{\partial x_j}, \frac{\partial^2 y_k}{\partial x_j^2}, \ldots, \frac{\partial^n y_k}{\partial x_j^n}) = 0 \qquad (1)$$

if it maps the set of solutions to itself, i.e. if

$$f(\bar{x}_j, \bar{y}_k, \frac{\partial \bar{y}_k}{\partial \bar{x}_j}, \frac{\partial^2 \bar{y}_k}{\partial \bar{x}_j^2}, \ldots, \frac{\partial^n \bar{y}_k}{\partial \bar{x}_j^n}) = 0$$

when (1) holds (2)

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The procedure is initiated with the group G, a class of r-parameters a_i, i = 1,2,…,r of the form (see Moran and Gaggioli [7])

$$G:\begin{cases}\overline{x}_i = C^{x_i}(a_1,a_2,...,a_r)\,x_i + K^{x_i}(a_1,a_2,...,a_r)\;,\; i=1,2,...,s\\ \overline{y}_j = C^{y_j}(a_1,a_2,...,a_r)\,y_j + K^{y_j}(a_1,a_2,...,a_r)\;,\; j=1,2,...,m\end{cases} \quad (3)$$

where, r = s-1, C's and K's are real valued and at least differentiable in their real arguments $a_1, a_2,…,a_r$.

Transformations of the derivatives are obtained from G via chain-rule operations

$$\overline{S}_{\overline{x}_i} = (C^S / C^{x_i})S_{x_i},\;\; \overline{S}_{\overline{x}_i\overline{x}_j} = (C^S / C^{x_i}C^{x_j})S_{x_i x_j},... \quad (4)$$

where

$$S_{x_i} = \frac{\partial S}{\partial x_i}, S_{x_i x_j} = \frac{\partial^2 S}{\partial x_i \partial x_j},...$$

and S stands for y_k, $k=1,...,m$.

The auxiliary conditions also should be invariant under the group G.

The transformations generators correspond to the group G are

$$X_k = \sum_{i=1}^{s}(\alpha_{ik}x_i + \beta_{ik})\frac{\partial}{\partial x_i} + \sum_{j=1}^{m}(\alpha_{(j+s)k}\,y_j + \beta_{(j+s)k})\frac{\partial}{\partial y_j},\quad k=1,2,...,s-1 \quad (5)$$

where,

$$\alpha_{ik} = \frac{\partial C^{x_i}}{\partial a_k}(a_1^0,a_2^0,...,a_{s-1}^0)$$

$$\beta_{ik} = \frac{\partial K^{x_i}}{\partial a_k}(a_1^0,a_2^0,...,a_{s-1}^0)$$

$$\beta_{(j+s)k} = \frac{\partial C^{y_j}}{\partial a_k}(a_1^0,a_2^0,...,a_{s-1}^0)$$

$$\beta_{(j+s)k} = \frac{\partial K^{y_j}}{\partial a_k}(a_1^0,a_2^0,...,a_{s-1}^0) \quad (6)$$

The characteristic equations which give the solutions of the ODEs of m-dependent variables yj, j = 1,2,…,m are

$$\frac{dx}{\alpha_{11}x + \beta_{11}} = \frac{dy_1}{\alpha_{21}y_1 + \beta_{21}} = ... = \frac{dy_m}{\alpha_{(m+1)1}y_m + \beta_{(m+1)1}} \quad (7)$$

The characteristic equations which give the similarity transformations of the PDEs of two independent variables x_1, x_2 and m-dependent variables y_j, j = 1,2,…, m are

$$\frac{dx_1}{\alpha_{11}x_1 + \beta_{11}} = \frac{dx_2}{\alpha_{21}x_2 + \beta_{21}} = \frac{dy_1}{\alpha_{31}y_1 + \beta_{31}} = ... = \frac{dy_m}{\alpha_{(m+2)1}y_m + \beta_{(m+2)1}} \quad (8)$$

For a system of PDEs of s-independent variables, s≥2 and m-dependent variables y_j, j = 1,2,…,m.

Two ways can be used to reduce system this system to a system of ODEs

Apply the one-parameter group (s-1) times, each time will reduce the number of independent variables by one, i.e. after (s-1) times the number will reduce to one. The characteristic equations from which obtain the new independent/dependent variables are

$$\frac{dx_1}{\alpha_{11}x_1 + \beta_{11}} = \frac{dx_2}{\alpha_{21}x_2 + \beta_{21}} = ... = \frac{dx_s}{\alpha_{s1}x_s + \beta_{s1}} = \frac{dy_1}{\alpha_{41}y_1 + \beta_{41}} = \frac{dy_2}{\alpha_{51}y_2 + \beta_{51}} = ... = \frac{dy_m}{\alpha_{(m+s)1}y_m + \beta_{(m+s)1}} \quad (9)$$

The following relations give the new independent variables η_k, k = 1,2,…, s-1.

$$\frac{dx_1}{\alpha_{11}x_1 + \beta_{11}} = \frac{dx_i}{\alpha_{i1}x_i + \beta_{i1}},\quad i=2,3,...,s \quad (10)$$

The following relations give the new dependent variables F_j ($\eta_1, \eta_2,…, \eta_{s-1}$), j = 1,2,…,m

$$\frac{dx_i}{\alpha_{i1}x_i + \beta_{i1}} = \frac{dy_j}{\alpha_{(j+s)1}y_j + \beta_{(j+s)1}},\quad i\in\{1,2,...,s\},\; j=1,2,...,m \quad (11)$$

Obtain the similarity generator X by cancelling

$$\frac{\partial g}{\partial x_3}, \frac{\partial g}{\partial x_4}, ..., \frac{\partial g}{\partial x_s}$$

from the system

$$X_k g = 0,\;\; k=1,2,...,s-1 \quad (12)$$

where X_k given in (5). The similarity generator is

$$X=\sum_{i=1}^{2}[g_i(x_3,x_4,\ldots,x_s)x_i+h_i(x_3,x_4,\ldots,x_s)]\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}[R_j(x_3,x_4,\ldots,x_s)y_j+Q_j(x_3,x_4,\ldots,x_s)]\frac{\partial}{\partial y_j} \quad (13)$$

where g_i, h_i, R_j and Q_j are functions of the independent variables $x_3, x_4, \ldots, x_s$..

The characteristic equations correspond to the generator X which give the similarity transformations are

$$\frac{dx_1}{g_1x_1+h_1}=\frac{dx_2}{g_2x_2+h_2}=\frac{dx_3}{0}=\frac{dx_4}{0}=\ldots=\frac{dx_s}{0}=\frac{dy_1}{R_1y_1+Q_1}=\frac{dy_2}{R_2y_2+Q_2}=\ldots=\frac{dy_m}{R_my_m+Q_m} \quad (14)$$

From the first two fractions calculates the relation between the new similarity independent variable η and the independent variables x_i, $i=1,2,\ldots,s$.

From the first or the second fraction with a fraction contains y_j we can find the relations between the old dependent variables y_j, $j = 1,2,\ldots, m$ and the new dependent variables $F_j(\eta)$. Using the previous relations, the PDEs transform to ODEs.

Now, I present the general formulations of the exact solution of ODEs and the similarity transformations of PDEs of two/three-independent variables whose the invariant group are known. I call these formulations HAMAD formulations.

HAMAD FORMULATIONS

Hamad Formulations for Exact Solutions of ODEs of m-Dependent Variables: In this section, I present the general formulations which give the exact solutions of the ODEs of the independent variable x and m-dependent variables y_j, $j = 1,2,\ldots, m$. From the characteristic equations (7), the formulations depend on the values of α_{i1} and β_{i1}, where $i = 1,2,\ldots,m+1$. Five formulations are found which as following.

(I1) If $\alpha_{11}\neq 0, \alpha_{(i+1)1}\neq 0, i=1,2,\ldots,m$

From Eq. (7), the formulation of the exact solutions are

$$y_i=\frac{k_i}{\alpha_{(i+1)1}}(\alpha_{11}x+\beta_{11})^{\frac{\alpha_{(i+1)1}}{\alpha_{11}}}-\frac{\beta_{(i+1)1}}{\alpha_{(i+1)1}} \quad (15)$$

(I2) If $\alpha_{11}\neq 0, \alpha_{(i+1)1}=0, \beta_{(i+1)1}\neq 0, i=1,2,\ldots,m$

Similarly, for this case the exact solutions are

$$y_i=\ln\left|k_i(\alpha_{11}x+\beta_{11})^{\frac{\beta_{(i+1)1}}{\alpha_{11}}}\right| \quad (16)$$

(I3) If $\alpha_{11}=0, \alpha_{(i+1)1}=0, \beta_{11}\neq 0, \beta_{(i+1)1}\neq 0, i=1,2,\ldots,m$

$$y_i=\frac{\beta_{11}}{\beta_{(i+1)1}}x+k_i \quad (17)$$

(I4) If $\alpha_{11}=0, \alpha_{(i+1)1}\neq 0, \beta_{11}\neq 0, i=1,2,\ldots,m$

$$y_i=\frac{k_i}{\alpha_{(i+1)1}}\exp\left(\frac{\alpha_{(i+1)1}}{\beta_{11}}x\right)-\frac{\beta_{(i+1)1}}{\alpha_{(i+1)1}} \quad (18)$$

(I5) If $(\alpha_{11}\neq 0 \text{ or } \beta_{11}\neq 0), \alpha_{(i+1)1}=0, \beta_{(i+1)1}=0, i=1,2,\ldots,m$

$$y_i=k_i \quad (19)$$

where, k_i, $i = 1,2,\ldots,m$ are constants. The constants α_1, α_{i+1}, β_1, β_{i+1}, k_i, $i = 1,2,\ldots,m$ will be determined to y_j satisfy the ODEs.

HAMAD Formulations for PDEs of Two Independent Variables: Consider PDEs of two independent variable x_1, x_2 and n-dependent variables y_j, $j = 1,2,\ldots, m$ *From the characteristic equations* (8) the similarity transformations are given from the following formulations which depend on the values of α_{i1}, β_{i1}, $i = 1,2,\ldots, m+2$. Five formulations for the similarity variables are found.

(II1) If $\alpha_{11}\neq 0, \alpha_{21}\neq 0$

The new independent variable η (similarity independent variable) is

$$\eta=\frac{(\alpha_{21}x_2+\beta_{21})^{\alpha_{11}}}{(\alpha_{11}x_1+\beta_{11})^{\alpha_{21}}}$$

or

$$\eta=\frac{\alpha_{21}x_2+\beta_{21}}{(\alpha_{11}x_1+\beta_{11})^{\alpha_{21}/\alpha_{11}}} \quad (20)$$

and the dependent variables y_j have the following formulae,

(i) For $\alpha_{(j+2)1}\neq 0, j=1,2,\ldots,m$

Also from Eq. (8) the relations between the original dependent variables y_j and the new dependent variables $F_j(\eta)$ are

$$y_j=\frac{1}{\alpha_{(j+2)1}}(\alpha_{k1}x_k+\beta_{k1})^{\frac{\alpha_{(j+2)1}}{\alpha_{k1}}}F_j(\eta)-\frac{\beta_{(j+2)1}}{\alpha_{(j+2)1}},\ k=1,2 \quad (21)$$

(ii) For $\alpha_{(j+2)1} = 0, \beta_{(j+2)1} \neq 0,\ j = 1,2,...,m$

$$y_j = \ln\left|(\alpha_{k1} x_k + \beta_{k1})^{\frac{\beta_{(j+2)1}}{\alpha_{k1}}} F_j(\eta)\right|,\ k = 1,2 \tag{22}$$

(II2) If $\alpha_{11} = \alpha_{21} = 0,\ \beta_{11} \neq 0, \beta_{21} \neq 0$

For this case the new independent variable has the following form

$$\eta = \beta_{21} x_1 - \beta_{11} x_2 \tag{23}$$

The dependent variables are as following

(i) For $\alpha_{(j+2)1} \neq 0, j = 1,2,...,m$

$$y_j = \frac{1}{\alpha_{(j+2)1}} \exp\left(\frac{\alpha_{(j+2)1}}{\beta_{k1}} x_k\right) F_j(\eta) - \frac{\beta_{(j+2)1}}{\alpha_{(j+2)1}},\ k = 1,2 \tag{24}$$

(ii) For $\alpha_{(j+2)1} = 0,\ \beta_{(j+2)1} \neq 0,\ j = 1,2,...,m$

$$y_j = F_j(\eta) + \frac{\alpha_{(i+2)1}}{\beta_{k1}} x_k,\quad k = 1,2 \tag{25}$$

(II3) If $\alpha_{11} = 0, \alpha_{21} \neq 0, \beta_{11} \neq 0$

The new independent variable corresponds to this case is

$$\eta = (\alpha_{21} x_2 + \beta_{21})^{\beta_{11}} \exp(-\alpha_{21} x_1) \tag{26}$$

For the dependent variables we have

(i) for $\alpha_{(j+2)1} \neq 0, j = 1,2,...,m$

y_j = given in Eq. (21) with k = 2 or y_j = given in Eq. (24) with k = 1

(ii) $\alpha_{(j+2)1} = 0,\ \beta_{(j+2)1} \neq 0,\ j = 1,2,...,m$

y_j = given in Eq. (22) with k = 2 or y_j = given in Eq. (25) with k = 1

(II4) If $\alpha_{11} = 0,\ \beta_{11} = 0, (\alpha_{21} \neq 0 \text{ or } \beta_{21} \neq 0)$

The similarity independent variable is

$$\eta = x_1 \tag{27}$$

The dependent variables are as following

(i) For $\alpha_{(j+2)1} \neq 0 \Rightarrow \begin{cases} \alpha_{21} \neq 0 \rightarrow y_j \\ \quad = \text{given in Eq.}(21) \text{ with } k = 2 \\ \alpha_{21} = 0, \beta_{21} \neq 0 \rightarrow y_j \\ \quad = \text{given in Eq.}(24) \text{ with } k = 2 \end{cases}$

(i) For

$$\alpha_{(j+2)1} = 0, \beta_{(j+2)1} \neq 0 \Rightarrow \begin{cases} \alpha_{21} \neq 0 \rightarrow y_j \\ \quad = \text{given in Eq.}(22) \text{ with } k = 2 \\ \alpha_{21} = 0, \beta_{21} \neq 0 \rightarrow y_j \\ \quad = \text{given in Eq.}(25) \text{ with } k = 2 \end{cases}$$

(II5) If $\alpha_{(j+2)1} = \beta_{(j+2)1} = 0,\ j = 1,2,...,m$

The dependent variables are

$$y_j = F_j(\eta) \tag{28}$$

where, η takes one from the previous forms.

The constants α_{j1} and α_{j1}, j = 1,2,…, m+2 will be chosen to get the full similarity representations.

HAMAD Formulations for PDEs of 3-Independent Variables: Two Steps to get the similarity transformations for system of partial differential equations contain three independent variables. Consider x_i, i = 1,2,3 are the independent variables and y_j, j = 1,2,…, m are the dependent variables.

Step 1: From equation (10) with i = 2,3 the new two independent variables η_1, η_2 are

$$\eta_{i-1} = \frac{(\alpha_{i1} x_i + \beta_{i1})^{\alpha_{11}}}{(\alpha_{11} x_1 + \beta_{11})^{\alpha_{i1}}}$$

or

$$\eta_{i-1} = \frac{\alpha_{i1} x_i + \beta_{i1}}{(\alpha_{11} x_1 + \beta_{11})^{\alpha_{i1}/\alpha_{11}}},\ i = 2,3 \tag{29}$$

Using the equation (11) can obtains the relations for the dependent variables y_j, j = 1,2,…, m which take the same formulations in section 4, but instead of k = 1,2 writes k = 1,2,3 and instead of $F_j(\eta)$ writes $F_j(\eta_1,\eta_2)$.

Step 2: From equation (5), the transformations generators for this case are

$$X_k = \sum_{i=1}^{3} (\alpha_{ik} x_i + \beta_{ik}) \frac{\partial}{\partial x_i} + \sum_{j=1}^{m} (\alpha_{(j+3)k} y_j + \beta_{(j+3)k}) \frac{\partial}{\partial y_j},\quad k = 1,2 \tag{30}$$

By deleting $\partial g/\partial x_1$ from the two equations $Xkg = 0$, $k = 1,2$, where g is a function of all independent and dependent variables. Then the similarity generator is

$$X=\sum_{i=2}^{3}[(\lambda_{1i21}x_1+\rho_{i112})x_i+\rho_{1i21}x_1+\sigma_{1i21}]\frac{\partial}{\partial x_i}+\sum_{j=1}^{m}\begin{bmatrix}(\lambda_{1(j+3)21}x_1+\rho_{(j+3)112})y_j\\+\rho_{1(j+3)21}x_1+\sigma_{1(j+3)21}\end{bmatrix}\frac{\partial}{\partial y_j} \quad (31)$$

where,

$$\begin{aligned}\lambda_{ijmn}&=\alpha_{im}\alpha_{jn}-\alpha_{in}\alpha_{jm}\\ \rho_{ijmn}&=\alpha_{im}\beta_{jn}-\alpha_{in}\beta_{jm}\\ \sigma_{ijmn}&=\beta_{im}\beta_{jn}-\beta_{in}\beta_{jm}\end{aligned} \quad (32)$$

The characteristic equations of (31) are

$$\begin{aligned}\frac{dx_1}{0}&=\frac{dx_2}{(\lambda_{1221}x_1+\rho_{2112})x_2+\rho_{1221}x_1+\sigma_{1221}}\\&=\frac{dx_3}{(\lambda_{1321}x_1+\rho_{3112})x_3+\rho_{1321}x_1+\sigma_{1321}}\\&=\frac{dy_1}{(\lambda_{1421}x_1+\rho_{4112})y_1+\rho_{1421}x_1+\sigma_{1421}}\\&=\frac{dy_2}{(\lambda_{1521}x_1+\rho_{5112})y_1+\rho_{1521}x_1+\sigma_{1521}}=\ldots\\&=\frac{dy_n}{(\lambda_{1(n+3)21}x_1+\rho_{(n+3)112})y_n+\rho_{1(n+3)21}x_1+\sigma_{1(n+3)21}}\end{aligned} \quad (33)$$

from which I introduce HAMAD formulations for the new similarity variables (independent and dependent). The new independent variable η (x_1, x_2, x_3) and the dependent variables forms depend on the values of λ_{ijmn}, ρ_{ijmn} and σ_{ijmn} as following

(III1) If $(\lambda_{1j21}\neq 0 \text{ or } \rho_{j112}\neq 0), j=2,3$

The similarity independent variables is

$$\eta=\frac{[(\lambda_{1321}x_1+\rho_{3112})x_3+\rho_{1321}x_1+\sigma_{1321}]^{(\lambda_{1221}x_1+\rho_{2112})}}{[(\lambda_{1221}x_1+\rho_{2112})x_2+\rho_{1221}x_1+\sigma_{1221}]^{(\lambda_{1321}x_1+\rho_{3112})}}$$

or

$$\eta=\frac{[(\lambda_{1321}x_1+\rho_{3112})x_3+\rho_{1321}x_1+\sigma_{1321}]}{\left[(\lambda_{1221}x_1+\rho_{2112})x_2+\rho_{1221}x_1+\sigma_{1221}\right]^{(\lambda_{1321}x_1+\rho_{3112})/(\lambda_{1221}x_1+\rho_{2112})}} \quad (34)$$

The formulations of the dependent variables y_j depend on the following cases

(i) If $(\lambda_{1(j+3)21}\neq 0 \text{ or } \rho_{(j+3)112}\neq 0)$, $j=1,2,\ldots,m$

$$y_j=\frac{[(\lambda_{1k21}x_1+\rho_{k112})x_k+\rho_{1k21}x_1+\sigma_{1k21}]^{(\lambda_{1(j+3)21}x_1+\rho_{(j+3)112})/(\lambda_{1k21}x_1+\rho_{k112})}}{\lambda_{1(j+3)21}x_1+\rho_{(j+3)112}}F_j(\eta)-\frac{\rho_{1(j+3)21}x_1+\sigma_{1(j+3)21}}{\lambda_{1(j+3)21}x_1+\rho_{(j+3)112}},\quad k=2,3 \quad (35)$$

(ii) If $\lambda_{1(j+3)21}=\rho_{(j+3)112}=0$, $(\rho_{1(j+3)21}\neq 0 \text{ or } \sigma_{1(j+3)21}\neq 0), j=1,2,\ldots,m$

$$y_j=\ln\left|F_j(\eta)[(\lambda_{1k21}x_1+\rho_{k112})x_k+\rho_{1k21}x_1+\sigma_{1k21}]^{(\rho_{1(j+3)21}x_1+\sigma_{1(j+3)21})/(\lambda_{1k21}x_1+\rho_{k112})}\right| \quad (36)$$

where, $k=2, 3$.

(III2) If $\lambda_{1221}=\rho_{2112}=0$, $(\rho_{1221}\neq 0 \text{ or } \sigma_{1221}\neq 0),(\lambda_{1321}\neq 0 \text{ or } \rho_{3112}\neq 0)$

For this case the dependent variable is

$$\eta=[(\lambda_{1321}x_1+\rho_{3112})x_3+\rho_{1321}x_1+\sigma_{1321}]\exp\left(\frac{-\lambda_{1321}x_1-\rho_{3112}}{\rho_{1221}x_1+\sigma_{1221}}\right) \quad (37)$$

and the independent variables are

(i) If $(\lambda_{1(j+3)21} \neq 0$ or $\rho_{(j+3)112} \neq 0)$, $j = 1,2,...,m$
y_j = given in Eq. (21) with k = 3

or

$$y_j = \frac{F_j(\eta)}{\lambda_{1(j+3)21} x_1 + \rho_{1(j+3)21}} \exp\left(\frac{\lambda_{1(j+3)21} x_1 + \rho_{1(j+3)21}}{\rho_{1221} x_1 + \sigma_{1221}}\right) - \frac{\rho_{1(j+3)21} x_1 + \sigma_{1(j+3)21}}{\lambda_{1(j+3)21} x_1 + \rho_{1(j+3)21}} \quad (38)$$

(ii) If $\lambda_{1(j+3)21} = \rho_{(j+3)112} = 0$, $(\rho_{1(j+3)21} \neq 0$ or $\sigma_{1(j+3)21} \neq 0)$, $j = 1,2,...,m$

y_j = given in Eq. (22) with k = 3

or

$$y_j = F_j(\eta) + \frac{\rho_{1(j+3)21} x_1 + \sigma_{1(j+3)21}}{\rho_{1221} x_1 + \sigma_{1221}} \quad (38)$$

(III3)If

$\lambda_{1221} = \rho_{1221} = \rho_{2112} = \sigma_{1221} = 0$,
$(\lambda_{1321} \neq 0$ or $\rho_{1321} \neq 0$ or $\rho_{3112} \neq 0$ or $\sigma_{1321} \neq 0)$

For this case, the similarity independent variable is

$$\eta = x_2 \quad (39)$$

The dependent variables are

(i) If $(\lambda_{1321} \neq 0$ or $\rho_{3112} \neq 0)$

Here, two cases are found

(a) $(\lambda_{1(j+3)21} \neq 0$ or $\rho_{(j+3)112} \neq 0)$, $j = 1,2,...,m$

y_j = given in Eq. (21) with k = 3

(b) $\lambda_{1(j+3)21} = \rho_{(j+3)112} = 0$, $(\rho_{1(j+3)21} \neq 0$ or $\sigma_{1(j+3)21} = 0)$, $j = 1,2,...,m$

y_j = given in Eq. (22) with k = 3

(ii) If $\lambda_{1321} = \rho_{3112} = 0$, $(\rho_{1321} \neq 0$ or $\sigma_{1321} \neq 0)$

Also, two cases are obtained
(a) $(\lambda_{1(j+3)21} \neq 0$ or $\rho_{(j+3)112} \neq 0)$, $j = 1,2,...,m$

y_j = given in subsection (III2)(i) but instead of ρ_{1221} and σ_{1221} write ρ_{1321} and σ_{1321} respectively.

(b) $\lambda_{1(j+3)21} = \rho_{(j+3)112} = 0$, $(\rho_{1(j+3)21} \neq 0$ or $\sigma_{1(j+3)21} \neq 0)$, $j = 1,2,...,m$

y_j = given in subsection (III2)(ii) but instead of ρ_{1221} and σ_{1221} write ρ_{1321} and σ_{1321} respectively.

(III4) If

$\lambda_{1(j+3)21} = \rho_{1(j+3)21} = \rho_{(j+3)112} = \sigma_{1(j+3)21} = 0$, $j = 1,2,...,m$

$$y_j = F_j(\eta) \quad (40)$$

where η is one of the previous formulations in subsections (29), (30) and (31).

EXAMPLES

Example 1: Consider the first-order ODE

$$\frac{dy}{dx} = y \quad (41)$$

has the following one parameter group G.

$$G: \begin{cases} \bar{x} = C^x(a_1)\, x + K^x(a_1) \\ \bar{y} = C^y(a_1)\, y + K^y(a_1) \end{cases} \quad (42)$$

and the transformation of the derivative is as follows (using chain-rule)

$$\frac{d\bar{y}}{d\bar{x}} = \frac{C^y(a_1)}{C^x(a_1)} \frac{dy}{dx} \quad (43)$$

The invariant condition is

$$\frac{d\bar{y}}{d\bar{x}} - \bar{y} = 0$$

when

$$\frac{dy}{dx} - y = 0 \quad (44)$$

From Eqs. (42) and (43) then

$$\frac{d\bar{y}}{d\bar{x}} - \bar{y} = \left[\frac{C^y}{C^x}\right]\frac{dy}{dx} - (C^y y + K^y) \quad (45)$$

Equation (45) satisfies the invariant condition (44) if

$$\frac{C^y}{C^x} = C^y, \qquad K^y = 0 \tag{46}$$

Then

$$C^x = 1, \qquad K^y = 0$$

Then, the group G becomes

$$G: \begin{cases} \bar{x} = x + K^x(a_1) \\ \bar{y} = C^y(a_1)\, y \end{cases} \tag{47}$$

From Eq. (6) then

$$\alpha_{11} = 0, \quad \beta_{21} = 0 \tag{48}$$

From the formulation (18) with $y_1 = y$, then

$$y_1 = y = \frac{k_1}{\alpha_{21}} \exp(\frac{\alpha_{21}}{\beta_{11}} x) \tag{49}$$

Substituting from (49) into (41), we get $\alpha_{21} = \beta_{11}$. Then the exact solution of Eq. (41) is

$$y = c\ e^x, \qquad c = k_1 / \alpha_{21} \tag{50}$$

Example 2: Consider the following first-order ODE of two dependent variables

$$\frac{dy}{dx} + zy + \frac{dz}{dx} = 0 \tag{51}$$

Similarly, the one parameter group G has the following form

$$G: \begin{cases} \bar{x} = C^x(a_1)\, x + K^x(a_1) \\ \bar{y} = (1/C^x(a_1))\, y \\ \bar{z} = (1/C^x(a_1))\, z \end{cases} \tag{52}$$

From (6) with consider $y_1 = y$ and $y_2 = z$, then

$$\beta_{21} = \beta_{31} = 0, \ \ \alpha_{21} = \alpha_{31} = \frac{-\alpha_{11}}{m} \tag{53}$$

where, $m = [C^x(a_1{}^0)]^2$. From the formulation (15) then

$$\begin{aligned} y_1 = y &= \frac{-m k_1}{\alpha_1} (\alpha_1 x + \beta_1)^{-1/m} \\ y_2 = z &= \frac{-m k_2}{\alpha_1} (\alpha_1 x + \beta_1)^{-1/m} \end{aligned} \tag{54}$$

where, k_1, k_2 and m are constants. Substituting from (54) into (51), we get

$$m = 1, \ k_1 = \frac{-k_2 (\alpha_1)^2}{k_2 + (\alpha_1)^2} \tag{55}$$

Hence, the solution of (51) is

$$\begin{aligned} y &= \frac{k_2 \alpha_1}{k_2 + (\alpha_1)^2} (\alpha_1 x + \beta_1)^{-1} \\ z &= \frac{-k_2}{\alpha_1} (\alpha_1 x + \beta_1)^{-1} \end{aligned} \tag{56}$$

Example 3: Consider the third-order ODE

$$\frac{d^3 y}{dx^3} + \frac{1}{2} y \frac{d^2 y}{dx^2} = 0 \tag{57}$$

has the one parameter group G:

$$G: \begin{cases} \bar{x} = C^x(a_1)\, x + K^x(a_1) \\ \bar{y} = (1/C^x(a_1))\, y \end{cases} \tag{58}$$

and from Eq. (6) we give

$$\beta_{21} = 0, \ \ \alpha_{21} = \frac{-\alpha_{11}}{m}, \quad m = (C^x(a_1{}^0))^2 \tag{59}$$

Using the formulation (15), then

$$y = \frac{-k_1 m}{\alpha_{11}} (\alpha_{11} x + \beta_{11})^{-1/m} \tag{60}$$

From (60) and (57), then Eq. (60) holds the ODE (57) if m = 1 and $k_1 = -6(\alpha_{11})^2$. Then the solution of (57) is

$$y = \frac{6\alpha_{11}}{\alpha_{11} x + \beta_{11}} \tag{61}$$

Example 4: Consider the one dimension heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{62}$$

Similarly, the PDE (62) is an invariant under the following group G

$$G: \begin{cases} \bar{x} = C^x(a_1)\, x + K^x(a_1) \\ \bar{t} = (C^x(a_1))^2\, t + K^t(a_1) \\ \bar{u} = C^u(a_1)\, u + K^u(a_1) \end{cases} \tag{63}$$

Assume, $x_1 = t$, $x_2 = x$ and $y_1 = u$, then from Eq. (6) we get

$$\alpha_{11} = 2m\alpha_{21}, \quad m = C^x(a_1^0) \tag{64}$$

From the formulations (20) and (21), then

$$\eta = \frac{(\alpha_{21}x + \beta_{21})^{2m\alpha_{21}}}{(2m\alpha_{21}t + \beta_{11})^{\alpha_{21}}} \tag{65}$$

$$y_1 = u = \frac{1}{\alpha_{31}}(\alpha_{k1}x_k + \beta_{k1})^{\alpha_{31}/\alpha_{k1}}F_1(\eta) - \frac{\beta_{31}}{\alpha_{31}}, \quad k = 1,2$$

(i) for k = 1

$$u = \frac{1}{\alpha_{31}}(2m\alpha_{21}t + \beta_{11})^{\alpha_{31}/(2m\alpha_{21})}F_1(\eta) - \frac{\beta_{31}}{\alpha_{31}} \tag{67}$$

Substituting from (65) and (67) into (62), then m = 1 is the value that give the full similarity representation. Then, Eq. (62) transforms to the following ODE (similarity representation)

$$4\alpha\eta^{(2-1/\alpha_{21})}F_1'' + 2\alpha\eta^{(1-1/\alpha_{21})}\left(2-\frac{1}{\alpha_{21}}\right)F_1' + \frac{2}{(\alpha_{21})^2}\eta F_1' - \frac{\alpha_{31}}{(\alpha_{21})^4}F_1 = 0 \tag{68}$$

Where primes in (68) denote to differentiation with respect to η and $\alpha_{21} = \beta_{31}$ are nonzero arbitrary constants.

If we put for example $\alpha_{21} = 1/2$ and $\beta_{31} = 1$, the similarity representation of (62) becomes

$$\alpha F_1'' + 2\eta F_1' - 4F_1 = 0 \tag{69}$$

(ii) For k = 2

$$u = \frac{1}{\alpha_{31}}(\alpha_{21}x + \beta_{21})^{\alpha_{31}/\alpha_{21}}F_1(\eta) - \frac{\beta_{31}}{\alpha_{31}} \tag{70}$$

Similarly, m = 1 gives the complete similarity representation. Then using (65) and (70) the PDE (62) transforms to the following ODE

$$4\alpha(\alpha_{21})^4\eta^{\frac{\alpha_{31}(\alpha_{21}-1)+4(\alpha_{21})^3}{2(\alpha_{21})^2}}F_1'' + 2\alpha(\alpha_{21})^2(2\alpha_{31} + 2(\alpha_{21})^2 - \alpha_{21})\eta F_1' + 2(\alpha_{21})^2\eta^{1+\frac{1}{\alpha_{21}}}F_1' + 2\alpha_{31}(\alpha_{31} - \alpha_{21})F_1 = 0 \tag{71}$$

where, α_{21} and β_{31} are nonzero arbitrary constants.

Example 5: Consider the following second-order PDE.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial y^2} \tag{72}$$

with the boundary conditions

$$u = 1 \quad \text{at } y = 0,\ t > 0 \tag{73}$$
$$u \to 0 \quad \text{as } y \to \infty,\ t > 0 \tag{66}$$

Using the invariant condition (2), I have gotten the PDE (72) and the boundary conditions (73) are invariant under the following two parameters group

$$G: \begin{cases} \bar{t} = \dfrac{1}{(C^y(a_1, a_2))^2}t + K^t(a_1, a_2) \\ \bar{x} = \dfrac{1}{(C^y(a_1, a_2))^2}x + K^x(a_1, a_2) \\ \bar{y} = C^y(a_1, a_2)y \\ \bar{u} = u \end{cases} \tag{74}$$

By considering $t = x_1$, $x = x_2$, $y = x_3$ and $u = y_1$, then from (6) we get

$$\begin{aligned} &\alpha_{41} = \alpha_{42} = \beta_{31} = \beta_{41} = \beta_{32} = \beta_{42} = 0, \\ &\alpha_{11} = \alpha_{21} = -\frac{2}{m^3}\alpha_{31}, \\ &\alpha_{12} = \alpha_{22} = -\frac{2}{m^3}\alpha_{32}, \\ &m = C^y(a_1^0, a_2^0) \end{aligned} \tag{75}$$

Substituting from (75) into (32) then

$$\begin{aligned} \lambda_{1321} &= \lambda_{1221} = \lambda_{1421} = \rho_{1321} = \rho_{1421} \\ &= \rho_{4112} = \sigma_{1321} = \sigma_{1421} = 0, \\ \rho_{3112} &= \frac{-m^3}{2}\rho_{2112} \end{aligned} \tag{76}$$

Substituting from (76) into the formulations (34) and (40), then the new independent variable and the dependent variable (similarity transformations) are

$$\eta = y(\rho_{2112}x + \rho_{1221}t + \sigma_{1221})^{\frac{m^3\rho_{2112}}{2}}, \quad u = F_1(\eta) \tag{77}$$

From (72), (73) and (76), we get m = -1, $\rho_{2112} = 1$ give the complete similarity representation. Then the similarity transformations become

$$\eta = \frac{y}{\sqrt{x + \rho_{1221} t + \sigma_{1221}}}, \quad u = F_1(\eta) \tag{78}$$

where, ρ_{1221} and σ_{1221} are arbitrary constants. The transformations (78) map Eq. (72) and the boundary condition (73) to the following ODE

$$F_1'' + \frac{1}{2}\eta(F_1 + \rho_{1221}) F_1' = 0 \tag{79}$$

with the boundary conditions

$$F_1(0) = 1, \quad F_1(\infty) = 0 \tag{80}$$

Example 6: Consider the second-order PDE of three independent variables

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} \tag{81}$$

Similarly, Eq. (81) is invariant under the following one parameter group

$$G_1 : \begin{cases} \bar{t} = (C^y(a_1))^2 t + K^t(a_1) \\ \bar{x} = (C^y(a_1))^2 x + K^x(a_1) \\ \bar{y} = C^y(a_1) y + K^y(a_1) \\ \bar{u} = u \\ \bar{v} = (1/C^y(a_1))v \end{cases} \tag{82}$$

Consider $x_1 = x$, $x_2 = t$, $x_3 = y$, $y_1 = u$ and $y_2 = v$. From (29) the new two independent variables are

$$\eta_1 = \frac{\alpha_{11} t + \beta_{21}}{\alpha_{11} x + \beta_{11}}, \quad \eta_2 = \frac{\alpha_{31} y + \beta_{31}}{(\alpha_{11} x + \beta_{11})^{\alpha_{31}/\alpha_{11}}} \tag{83}$$

From (21) and (28) with replace $F_1(\eta)$, $F_2(\eta)$ with $F_1(\eta_1, \eta_2)$, $F_1(\eta_2, \eta_2)$ then the dependent variables take the following formulations

$$u = F_1(\eta_1, \eta_2), \quad v = (\alpha_{11} x + \beta_{11})^{-4(\alpha_{31}/\alpha_{11})^3} F_2(\eta_1, \eta_2) \tag{84}$$

Substituting from (83) and (84) into (81), we reduce the independent variables to two and the PDE (81) becomes

$$\frac{\partial^2 F_1}{\partial \eta_2^2} + 4(\eta_1 F_1 - 1)\frac{\partial F_1}{\partial \eta_1} + 2(\eta_2 F_1 - F_2)\frac{\partial F_1}{\partial \eta_2} = 0 \tag{85}$$

Eq. (85) is invariant under the following one parameter group

$$G_2 : \begin{cases} \bar{\eta}_1 = (C^{\eta_2}(a))^2 \eta_1 \\ \bar{\eta}_2 = C^{\eta_2}(a)\eta_2 \\ \bar{F}_1 = \dfrac{1}{(C^{\eta_2}(a))^2} F_1 \\ \bar{F}_2 = \dfrac{1}{C^{\eta_2}(a)} F_2 \end{cases} \tag{86}$$

Using (6) with the formulations (20) and (21), the similarity independent (η) and dependent ($G_1(\eta)$, $G_2(\eta)$) variables are given as following

$$\eta = \alpha_{21}\eta_2(2m\alpha_{21}\eta_1)^{\frac{-1}{2m}} \text{ (without lose of generality)} \Rightarrow \eta = \eta_2(\eta_1)^{\frac{-1}{2m}} \tag{87}$$

$$F_1 = \frac{-m^3}{2\alpha_{21}}(2m\alpha_{21}\eta_1)^{\frac{-1}{m^4}} G_1(\eta) \text{ (without lose of generality)} \Rightarrow F_1 = (\eta_1)^{\frac{-1}{m^4}} G_1(\eta) \tag{88}$$

$$F_2 = \frac{1}{\alpha_{21}}(2m\alpha_{21}\eta_1)^{\frac{-1}{2m^3}} G_2(\eta) \text{ (without lose of generality)} \Rightarrow F_2 = (\eta_1)^{\frac{-1}{2m^3}} G_2(\eta) \tag{89}$$

where, $m = C^{\eta 2}(a^0)$.

By substituting from (87), (88) and (89) into (85), the complete similarity representation is given when $m = 1$, as follows

$$G_1'' + 2(\eta - G_2) G_1' + 4(1 - G_1) G_1 = 0 \tag{90}$$

CONCLUSION

In this paper we have presented the following forms

- Formulations for the exact solutions of ODEs
- Formulations for similarity transformations for second dimensional PDEs
- Formulations for similarity transformations for third dimensional PDEs

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