World Applied Sciences Journal 12 (3): 354-357, 2011
ISSN 1818-4952

On Rotation Surfaces with Pointwise 1-Type Gauss Map in the 3-Dimensional Dual Space

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Abstract: In this paper, rotation surfaces with the pointwise 1-type Gauss map are studied in the 3-dimensional Dual space. By the use of the concept of pointwise finite type Gauss map, a characterisation theorem concerning rotation surfaces and constancy of the mean curvature of certain open subsets on these surfaces are obtained.

AMS Subject Classification: 53A17 · 53A25
Key words: Dual number · Dual vector · Gauss map · Rotation surface

INTRODUCTION

Dual numbers were introduced in the 19 th. century by Cilliford [1]. Dual quantities, the differential geometry of dual curves and application to the theoretical space kinematics were given by Veldkamp [2]. V. Brodsky and M. Shoham examined dual numbers representation of rigid body dynamics [3]. A. Parkın studied orthogonal matrix transformations [4] and Yang examined quaternion algebra and dual numbers [5]. Y.H. Kim and D.W. Yoon studied ruled surfaces with the pointwise 1-type Gauss map. They classified all submanifolds in an m-Euclidean space E^m satisfying the following equation:

$$\Delta G = f\,G \qquad (1.1)$$

Where Δ in the Laplacian of the induced metric and G the Gauss map for the submanifold, for some function f on the submanifold [6]. A Niang did studies on rotation surfaces in the minkowski 3-dimensional space with the pointwise 1-type Gauss map [7]. M. Choi and Y. H. Kim examined characterisation of the helicoid as ruled surfaces with pointwise 1-type Gauss map [8].

In this study, the condition (1.1) will be expressed in D^3, i.e,

$$\hat{G} = \hat{f}\hat{G} \qquad (1.2)$$

Where $\hat{\Delta} = \Delta + \varepsilon\Delta^*$ is the Laplacian in D^3, $\hat{G} = G + \varepsilon G^*$ is dual Gauss map, $\hat{f} = f + \varepsilon f^*$ is a dual funcition.

The Main Goal of this Article Is to Prove the Following Theorem

Theorem 1.1: Let $\hat{M}$ be a connected surface of rotation in a 3-dimensional Dual space, whose axis of rotation is $\hat{L}$. Let $\hat{M}'$ be any connected component of the subset $\hat{M} - \hat{L}$. Then $\hat{M}'$ is the pointwise 1-type Gauss map if and only if $\hat{M}'$ has a constant mean curvature.

Throughout this paper, we assume that all objects are smooth and all surfaces are connected unless otherwise mentioned in D^3.

Preliminaries: A dual number can be defined as an ordered pair combining a real part, a and a dual part a*,

$$\hat{a} = a + \varepsilon a^* \qquad (2.1)$$

Where ε is the dual unit with multiplication rule $\varepsilon^2 = 0$. An ordered triple of dual numbers $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ is called dual vector, we write $(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \hat{x}$. The numbers $\hat{x}_1, \hat{x}_2, \hat{x}_3$ are called the coordinates of $\hat{x}$.

Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and $\hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ be two dual vector, then inner product and cross-product of these two dual vectors are defined as follows;

$$\langle \hat{x}, \hat{y} \rangle = \hat{x}_1\hat{y}_1 + \hat{x}_2\hat{y}_2 + \hat{x}_3\hat{y}_3 \qquad (2.2)$$

$$\hat{x} \times \hat{y} = (\hat{x}_2 y_3 - \hat{x}_3\hat{y}_2,\ \hat{x}_3\hat{y}_1 - \hat{x}_1\hat{y}_3,\ \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1) \qquad (2.3)$$

If $\hat{x} \neq 0$ the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by $\langle \hat{x}, \hat{x} \rangle^{1/2}$.
A dual function of a dual space is given by,

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$$\hat{f}(\hat{t}) = f(t, t^*) + ef^*(t, t^*) \tag{2.4}$$

Where $\hat{t} = t + et^*$ is a dual variable, f and f* are two, generally different, functions of the two variables. This type of function is referred to simply as the dual functions throughout the paper.

Properties of dual functions were thoroughly investigated by H. Hacısalihoğlu [9]. He derived the general expression for dual analytic (differentiable) function as follows;

$$\hat{f}(t + et^*) = f + ef^* = f(t) + et^*f(t). \tag{2.5}$$

The analytic condition for a dual functions is

$$\frac{\partial f^*}{\partial t^*} = \frac{\partial f}{\partial t}. \tag{2.6}$$

The derivative of such a dual function with respect to a dual variable is

$$\frac{d\hat{f}(\hat{t})}{d\hat{t}} = \frac{\partial f}{\partial t} + e\frac{\partial f^*}{\partial t} = f' + et^*f''. \tag{2.7}$$

A dual function of two dual parameters is given by

$$\hat{F}(\hat{u},\hat{v}) = F(\hat{u},\hat{v}) + eF^*(\hat{u},\hat{v}). \tag{2.9}$$

Where $\hat{u} = u + eu^*$ and $\hat{v} = v + ev^*$ are two dual variables F and F* are two functions of two dual parameters. A dual analytic function of two dual parameters expressed as follows;

$$\hat{F}(\hat{u},\hat{v}) = F(\hat{u},\hat{v}) + eF^*(\hat{u},\hat{v}) = F(\hat{u},\hat{v}) + e(u^*F_u(u,v) + v^*Fv(u,v)) \tag{2.10}$$

The partial derivatives of Eq. (2.10) are given by

$$\hat{F}_{\hat{u}}(\hat{u},\hat{v}) = F_{\hat{u}}(\hat{u},\hat{v}) + eF_{\hat{u}}{}^*(\hat{u},\hat{v}) = F_u(u,v) + e(u^*F_{uu}(u,v) + v^*F_{vu}(u,v)) \tag{2.11}$$

$$\hat{F}_{\hat{v}}(\hat{u},\hat{v}) = F_{\hat{v}}(\hat{u},\hat{v}) + eF_{\hat{v}}{}^*(\hat{u},\hat{v}) == F_u(u,v) + e(u^*F_{uu}(u,v) + v^*F_{vu}(u,v)) \tag{2.11}$$

From this definition we are given some examples as follows;

$$\hat{F}(\hat{u},\hat{v}) = \sin(u+\epsilon u^*)\ \cos(v+\epsilon v^*) = \sin u\ \cos v + \epsilon(u^*\cos u \cos v - v^*\sin u \sin v)$$

$$\frac{\partial}{\partial \hat{u}}\hat{F}(\hat{u},\hat{v}) = \frac{\partial}{\partial \hat{u}}(\sin(u+\epsilon u^*)\cos(v+\epsilon v^*)) = \cos u\ \cos v + \epsilon(-u^*\sin u \cos v - v^*\cos u \sin v)$$

$$\frac{\partial}{\partial \hat{v}}\hat{F}(\hat{u},\hat{v}) = \frac{\partial}{\partial \hat{v}}(\sin'(u+\epsilon u^*)\cos'(v+\epsilon v^*)) = -\cos u\ \sin v + \epsilon(-u^*\cos u \sin v - v^*\sin u \cos v)$$

The Surface in D^3:
The surface $\hat{M}$ in D^3 is described locally by

$$\hat{X}:\hat{U} \subset D^2 \to D^3$$
$$(\hat{u},\hat{v}) \to \hat{X}(\hat{u},\hat{v}) = X(\hat{u},\hat{v}) + eX^*(\hat{u},\hat{v})$$
$$\hat{X}(\hat{u},\hat{v}) = X(u,v) + e(u^*X_u(u,v) + v^*X_v(u,v)) \tag{3.1}$$

Where $(\hat{u},\hat{v})$ are local coordinates on the open set $\hat{U}$ of D^2.

The Gauss map $\hat{G} = G + \varepsilon G^*$ on $\hat{U}$ is given by the following formulae:

$$\hat{G} = \frac{\hat{X}_{\hat{u}} \times \hat{X}_{\hat{v}}}{\|\hat{X}_{\hat{u}} \times \hat{X}_{\hat{v}}\|} = G + e(u^*G_u + v^*G_v)$$
$$G = \frac{X_u \times X_v}{\|X_u \times X_v\|} \tag{3.2}$$

The first and second fundamental forms $\hat{I} = I + eI^*$ and $\hat{II} = II + eII^*$, respectively, are obtained by

$$\hat{I} = \langle \hat{X}_{\hat{u}}, \hat{X}_{\hat{u}} \rangle (\hat{u}')^2 + 2\langle \hat{X}_{\hat{u}}, \hat{X}_{\hat{v}} \rangle \hat{u}'\hat{v}' + \langle \hat{X}_{\hat{v}}, \hat{X}_{\hat{v}} \rangle (\hat{v}')^2$$
$$\hat{I} = I + e(u^*I_u + v^*I_v)$$
$$I = \langle X_u, X_u \rangle (u')^2 + 2\langle X_u, X_v \rangle u'v' + \langle X_v, X_v \rangle (v')^2 \tag{3.3}$$

$$\hat{II} = \langle \hat{G}, \hat{X}_{\hat{u}\hat{u}} \rangle (\hat{u}')^2 + 2\langle \hat{G}, \hat{X}_{\hat{u}\hat{v}} \rangle \hat{u}'\hat{v}' + \langle \hat{G}, \hat{X}_{\hat{v}\hat{v}} \rangle (\hat{v}')^2$$
$$\hat{II} = II + e(u^*II_u + v^*II_v)$$
$$II = \langle G, X_{uu} \rangle (u')^2 + 2\langle G, X_{uv} \rangle u'v' + \langle G, X_{vv} \rangle (v')^2 \tag{3.4}$$

The mean curvature $\hat{H} = H + eH^*$ on $\hat{U}$ is given by the following formulae:

$$2\hat{H}=\frac{<\hat{G},\hat{X}_{\hat{u}\hat{u}}><\hat{X}_{\hat{u}},\hat{X}_{\hat{u}}>-2<\hat{G},\hat{X}_{\hat{u}\hat{v}}><\hat{X}_{\hat{u}},\hat{X}_{\hat{v}}>+<\hat{G},\hat{X}_{\hat{v}\hat{v}}><\hat{X}_{\hat{v}},\hat{X}_{\hat{v}}>}{<\hat{X}_{\hat{u}},\hat{X}_{\hat{u}}><\hat{X}_{\hat{v}},\hat{X}_{\hat{v}}>-(<\hat{X}_{\hat{u}},\hat{X}_{\hat{v}}>)^2}$$

$$2\hat{H}=2H+2e(u^*H_u+v^*H_v)$$

$$2H=\frac{<G,X_{uu}><X_u,X_u>-2<G,X_{uv}><X_u,X_v>+<G,X_{vv}><X_v,X_v>}{<X_u,X_u><X_v,X_v>-(<X_u,X_v>)^2} \tag{3.5}$$

Laplacian with respect to local coordinates $(\hat{x}_1,\hat{x}_2)$ on $\hat{U}$ for surface M is obtained as follows:

$$\hat{\Delta}=-\frac{1}{\sqrt{\det(\hat{g}_{ij})}}\sum\frac{\hat{\partial}}{\partial\hat{x}_i}\left(\sqrt{\det(\hat{g}_{ij})}\hat{g}^{ij}\frac{\hat{\partial}}{\partial\hat{x}_j}\right) \tag{3.6}$$

Where $(\hat{g}_{ij})$ is a dual matrix and $\hat{g}_{ij}=<\hat{X}_{\hat{x}_i},\hat{X}_{\hat{x}_j}>$, the dual matrix $(\hat{g}^{ij})$ is the inverse matrix of $(\hat{g}_{ij})$.

Rotation Surfaces in D^3:
The subgroup of rotations around the $\hat{z}$-axis consist of

$$\begin{bmatrix}\cos\hat{?} & -\sin\hat{?} & 0\\ \sin\hat{?} & \cos\hat{?} & 0\\ 0 & 0 & 1\end{bmatrix},\ \hat{?}=?+e?^* \tag{4.1}$$

A surface is called a surface of revolution if its image is stable under a 1-parameter subgroup of isometries which leaves a line pointwise fixed.

This general definition will be related to the ordinary one in terms of rotating a profile curve which lies in a certain plane containing the axis of rotation.

For a surface of revolution corresponding to an axis $\hat{L}$, let $\hat{M}'$ be any connected component of the subset $\hat{M}-\hat{L}$. We have the following lemma.

Lemma 4.1: If $\hat{L}$ is a rotation axis, then $\hat{M}'$ is expressed in the form $\hat{x}=\hat{r}(\hat{s})\,\mathrm{coc}\hat{?}$, $\hat{y}=\hat{r}(\hat{s})\sin\hat{?}$, $\hat{z}=\hat{h}(\hat{s})$; $\hat{?}=?+e?^*$, $\hat{s}=s+es^*$, with metric

$$\hat{I}=(\hat{r}'^2+\hat{h}'^2)d\hat{s}^2+\hat{r}^2d\hat{?}^2 \tag{4.2}$$

Where $\hat{r}(\hat{s})$ and $\hat{h}(\hat{s})$ are dual smooth functions of the parameter $\hat{s}$ such that $\hat{r}'^2+\hat{h}'^2=1$ and $\hat{r}(\hat{s})\neq 0$ for all $\hat{s}$.

Conversely, a surface given in the above form is a surface of revolution, the profile curve is $\hat{s}\to\hat{x}=\hat{r}(\hat{s}),\hat{z}=\hat{h}(\hat{s})$, where $\hat{s}$ is an arc parameter and

$$\hat{r}'^2+\hat{h}'^2=1 \tag{4.3}$$

In addition to above lemma we have the followings results.

Lemma 4.2: For the surface of revolution given in the Lemma 3.1 and expressed in the form

$$\hat{X}(\hat{s},\hat{\theta})=(\hat{r}(\hat{s})\cos\hat{\theta},\hat{r}(\hat{s})\sin\hat{\theta},\hat{h}(\hat{s})) \tag{4.4}$$

We have the following results.
The first and second fundamental forms are given by

$$\begin{aligned}&\hat{I}=d\hat{s}^2+\hat{r}d\hat{?}^2\\&I\hat{I}=(\hat{r}'\hat{h}''-\hat{h}'\hat{r}'')d\hat{s}^2+\hat{h}'\hat{r}d\hat{?}^2\end{aligned} \tag{4.5}$$

The mean curvature $\hat{H}$ satisfies

$$\begin{aligned}&2\hat{H}=(-\hat{r}''\hat{h}'+\hat{h}''\hat{r}')+\frac{\hat{h}'}{\hat{r}}\\&2\hat{H}'=(-\hat{r}'''\hat{h}'+\hat{h}'''\hat{r}')+\left(\frac{\hat{h}'}{\hat{r}}\right)'\end{aligned} \tag{4.6}$$

The Laplacian is given by

$$\hat{\Delta}=-\left[\frac{\hat{\partial}^2}{\partial\hat{s}^2}+\frac{\hat{r}'}{\hat{r}}\frac{\hat{\partial}}{\partial\hat{s}}+\frac{1}{\hat{r}^2}\frac{\hat{\partial}^2}{\partial\hat{?}^2}\right] \tag{4.7}$$

The Proof of the Theorem: We consider a surface of revolution $\hat{M}'$ in lemma 4.2. Then $\hat{M}'$ is a connected component of the set $\hat{M}-\hat{L}$. Let's express the condition $\hat{\Delta}\,\hat{G}=\hat{f}\hat{G}$ on $\hat{M}'$ for the Gauss map $\hat{G}=(-\hat{h}'\cos\hat{?},\,-\hat{h}'\sin\hat{?},\,\hat{r}')$.

We get from $\hat{G}$ the following three vectors

$$\begin{aligned}&\hat{G}_{\hat{s}}=(-\hat{h}''\cos\hat{?},-\hat{h}''\sin\hat{?},\hat{r}'')\\&\hat{G}_{\hat{s}\hat{s}}=(-\hat{h}'''\cos\hat{?},-\hat{h}'''\sin\hat{?},\hat{r}''')\\&\hat{G}_{\hat{?}\hat{?}}=(\hat{h}'\cos\hat{?},\hat{h}'\sin\hat{?},0)\end{aligned} \tag{5.1}$$

Then the Laplacian of the Gauss map by applying the formula (4.7) is the vector;

$$?\hat{G} = -[\hat{G}_{\hat{s}\hat{s}} + \frac{\hat{r}'}{\hat{r}}\hat{G}_{\hat{s}} + \frac{1}{\hat{r}^2}\hat{G}_{??}] \quad (5.2)$$

So we get easily that

$$\hat{\Delta}\hat{G} = ((E + \frac{\hat{r}'}{\hat{r}}\hat{h}'' - \frac{\hat{h}'}{\hat{r}^2})\cos\hat{?}, (\hat{h}''' + \frac{\hat{r}'}{\hat{r}}\hat{h}'' - \frac{\hat{h}'}{\hat{r}^2})\sin\hat{?}, -\hat{r}''' - \frac{\hat{r}'}{\hat{r}}\hat{r}'') \quad (5.3)$$

From this formula it is convenient to introduce the following functions. Let $\hat{A} = (\hat{h}''' + \frac{\hat{r}'}{\hat{r}}\hat{h}'' - \frac{\hat{h}'}{\hat{r}^2})$ and $\hat{B} = -\hat{r}''' - \frac{\hat{r}'}{\hat{r}}\hat{r}''$. Then we can write that

$$?\hat{G} = (\hat{A}\cos\hat{?}, \hat{A}\sin\hat{?}, \hat{B}) \quad (5.4)$$

So we have

$$<?\hat{G},\hat{G}> = -\hat{A}\hat{h}' + \hat{B}\hat{r}' \quad (5.5)$$

Since the condition $?\hat{G} = \hat{f}\hat{G}$ is equivalent to the condition

$$?\hat{G} - <?\hat{G}, \hat{G}> \hat{G} = 0 \quad (5.6)$$

This condition is then equivalent to following three equations:

$$\begin{aligned} &[\hat{A} + \hat{h}'(-\hat{A}\hat{h}' + \hat{B}\hat{r}')]\cos\hat{?} = 0 \\ &[\hat{A} + \hat{h}'(-\hat{A}\hat{h}' + \hat{B}\hat{r}')]\sin\hat{?} = 0 \\ &\hat{B} - \hat{r}'(-\hat{A}\hat{h}' + \hat{B}\hat{r}') = 0 \end{aligned} \quad (5.7)$$

These are equivalent to the two equations

$$\begin{aligned} &\hat{A}(1 - \hat{h}'^2) + \hat{B}\hat{r}\hat{h}' = 0 \\ &\hat{B}(1 - \hat{r}'^2) + \hat{A}\hat{h}'\hat{r}' = 0 \end{aligned} \quad (5.8)$$

Hence we obtain

$$\begin{aligned} &\hat{r}'(\hat{A}\hat{r}' + \hat{B}\hat{h}') = 0 \\ &\hat{h}'(\hat{A}\hat{r}' + \hat{B}\hat{h}') = 0 \end{aligned} \quad (5.9)$$

By (4.1), we get

$$\hat{A}\hat{r}' + \hat{B}\hat{h}' = (-\hat{r}'''h' + \hat{h}'''\hat{r}') + (\frac{\hat{h}'}{\hat{r}})' \quad (5.10)$$

On the other hand, for the derivative of mean curvature $\hat{H}$ from the second formula in (4.6) we get

$$\hat{r}'''\hat{h}' + \hat{r}'\hat{h}''' = 2\hat{H}' - (\frac{\hat{h}'}{\hat{r}})'. \quad (5.11)$$

Thus, from Eq. (5.10) and Eq. (5.11) we have

$$\hat{A}\hat{r}' + \hat{B}\hat{h}' = 2\hat{H}'. \quad (5.12)$$

Now the condition (5.9) becomes

$$\begin{aligned} &\hat{r}'\hat{H}' = 0 \\ &\hat{h}'\hat{H} = 0. \end{aligned} \quad (5.13)$$

s From this, we get with the Eq. (4.1) that $\hat{H}'$ is zero, therefore $\hat{H}'$ is identically zero on $\hat{M}'$. This proves the theorem.

Consequently 5.1: The surface of revolution with constant mean curvature in D^3 has screw motion.

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