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# The Irrationality of Fibonacci, Lucas and Pell Numbers

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**Abstract:** In this paper not only discussed the irrational behavior of the generalized Fibonacci numbers but also study the behavior of $\zeta(2)$ using Lucas and Pell numbers.

**Key words:** Fibonacci . lucas and pell numbers . pell equation

## INTRODUCTION

Fibonacci, Lucas and Pell numbers are discussed in various studies (Leveque, 1990; Duverney, 1993; Matala-Aho and Vanen 1998; Kono and Uehara, 1998; Shiokara 1998). The aim of this article is to observe irrational behavior of these numbers. This study also discussed the irrationality of $\zeta(2)$ by using q-derivative and q-integral.

## FIBONACCI NUMBERS

Let a Fibonacci number $F_n$ be defined as $F_1=F_2=1$, $F_n = F_{n-1} + F_{n-2}$, for $n\geq 3$. It is well known that Fibonacci numbers are relatively prime.
Therefore

- $F_{m+n} = F_{m-1}F_n + F_m F_{n+1}$
- $F_{mn}$ is divisible by $F_m$ for $m\geq 1, n\geq 1$
- If $m = qn+r$, then $\gcd(F_m, F_n) = \gcd(F_r, F_n)$
- $\gcd(F_m F_n)=F_d$ where $d = \gcd(m,n)$
- $\left(\frac{F_m}{F_n}\right)$ if and only if m/n for $m\geq n\geq 2$

**Lemma:** For $n\geq 1$, we have

- $F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1} \quad (n\geq 2)$
- $F_{2k}^2 = F_{2k+1}F_{2k-1} - 1$
- $F_1 + F_3 + F_5 + \cdots\cdots + F_{2n-1} = F_{2n}$
- $F_2 + F_4 + F_6 + \cdots\cdots + F_{2n} = F_{2n+1} - 1$
- $F_1 - F_2 + F_3 - F_4 + \cdots\cdots + (-1)^{n+1}F_n = 1 + (-1)^{n+1}F_{n-1}$
- $F_1^2 + F_2^2 + F_3^2 + \cdots\cdots + F_n^2 = F_n \times F_{n+1}$

Now define

$$\ell_q(x;s) = \sum_{n=1}^{\infty} \frac{x^n}{(1-q)^s} \text{ for } 0<|q|<1, 0<x<1$$

By taking

$$q = -\beta^2, x = -\beta^3 \left(\text{where } \beta = \frac{1-\sqrt{5}}{2}\right) \text{ and } \alpha\beta = -1$$

we have

$$(\alpha-\beta)^3 \ell_{-\beta^2}(-\beta^3;3) = \sum_{n=1}^{\infty} \frac{1}{F_n^3}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

So

$$\sum_{n=1}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

is irrational number. It is easy to see that

$$\ell_q(x;3) = x\frac{\sum_{n=1}^{\infty} q^n n e'_q(zq^n) \Pi_{\ell=1, n\neq \ell}^{\infty} e_q(zq^i)}{\Pi_{n=1}^{\infty}, e_q(zq^i)}$$

where

$$e_q(x) = \sum_{n=0}^{\infty} \frac{1}{(1-q)(1-q^2)\cdots\cdots(1-q^n)} x^n$$

for $|q|<1$; therefore we have the following theorem.

**Theorem 1:** $\sum_{n=1}^{\infty} \frac{1}{F_n^3}$ is an irrational number.

**Remark:** Let $\{F_n\}_{n\geq 1}$ be Fibonacci numbers. Then several results of Fibonacci zeta functions with Riemann zeta functions may be compared as under:

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Let $\{F_n\}_{n\geq 1}$ be Fibonacci numbers.

| | | |
|---|---|---|
| 1. | $\sum_{n=1}^{\infty}\frac{1}{F_{a^n+b}}, (a,b)\neq(2,0)$ is irrational (Kumiko Nishioka) | $\sum_{n=1}^{\infty}\frac{1}{a^n+b}, (a\geq 2), \forall b\in\mathbb{N}$ is irrational (Browein, 94) |
| 2. | $\sum_{n=1}^{\infty}\frac{1}{F_n}$ is irrational (Andre-Jeannin, Duverney) | |
| 3. | $\sum_{n=1}^{\infty}\frac{1}{F_n^{2k}}$ is transcendental for $k\in\mathbb{N}$ (D. Nishioka, I. Shiokawa) | $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\zeta(2k)$ is transecendental |
| 4. | $\sum_{n=1}^{\infty}\frac{1}{F_{a^n+b}^k}$ is transcendental $(a,b\in\mathbb{N},(a,b)\neq(2,0),k\geq 2)$ | $\sum_{n=1}^{\infty}\frac{1}{(a^n+b)^k}$ is unknown |
| 5. | $\sum_{n=1}^{\infty}\frac{1}{F_n^3}$ is irrational (I. Shiokara, T. Kim) | $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is irrational |
| 6. | $\sum_{n=1}^{\infty}\frac{(-1)^n}{F_nF_{n+1}}=\frac{1+\sqrt{5}}{2}$ | $\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+1)}=1-2\log 2$ |
| 7. | $\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+1}}$ is irrational | $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$ |
| 8. | $\sum_{n=1}^{\infty}\frac{1}{F_{2n}}$ is irrational | |
| 9. | $\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}$ is unknown | |
| 10. | $\sum_{n=1}^{\infty}\frac{1}{F_{2n}^2+1}$ is unknown | |
| 11. | $\sum_{n=1}^{\infty}\frac{1}{F_{n^2}}$ is unknown | |
| 12. | $\sum_{n=1}^{\infty}\frac{1}{(F_nF_{n+1})^k}$ is unknown (k≥2) | |

## LUCAS AND PELL NUMBERS

The Lucas numbers $F_n$ are defined by the same rule as the Fibonacci numbers as follows; $L_0 = 2$, $L_1 = 1$, $L_2 = 3$, $L_{n+2} = L_{n+1}$, $n\geq 1$, then Lucas and Fibonacci numbers are related to each other. For example,

- $F_{2n} = F_n L_n$
- $L_{2n} = L_n - 2(-1)^n$
- $\sum_{k=0}^{n} L_k = L_{n+2} - 1$
- $L_n = F_{n-1} + F_{n+1}$
- $5F_n = L_{n-1} + L_{n+1}$
- $2F_{m+n} = F_m L_n + F_n L_m$ and
- $2L_{m+n} = L_m L_n + 5F_m F_n$

Note that for algebraic numbers

$$\tau = \frac{1+\sqrt{5}}{2}$$

and

$$\sigma = \frac{1-\sqrt{5}}{2}$$

$$F_n = \frac{\tau^n - \sigma^n}{\tau - \sigma}$$

and we also know that

- $L_n = \tau^n + \sigma^n$,
- $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \lim_{n\to\infty}\frac{L_{n+1}}{L_n} = \tau$ and
- $\lim_{n\to\infty}\frac{L_n}{F_n} = \sqrt{5}$.

The Pell numbers $P_n$ and $Q_n$ are defined as

$$P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}\ (n\geq 2)$$

$$Q_0 = 0, Q_1 = 1, Q_n = 2Q_{n-1} + Q_{n-2}\ (n\geq 2)$$

which are also known as generalized Fibonacci numbers.

**Proposition 1:** If

$$\alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2}$$

are the roots of $x^2$-2x-1 = 0, then Pell numbers can be expressed as

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}},\ Q_n = \frac{\alpha^n + \beta^n}{2} \text{ for } n\geq 0$$

**Proof:** If $\alpha$ and $\beta$ are the roots of $x^2-2x-1=0$, so the results are obtained.

**Proposition 2:** For the Pell numbers when $n\geq 1$, we have the following relations:

- $P_{2n}=2P_nQ_n$
- $P_n+P_{n-1}=Q_n$
- $2Q_n^2-Q_{2n}=(-1)^{n+1}$
- $P_n+P_{n+1}+P_{n+3}=3P_{n+2}$ and
- $Q_n^2-2P_n^2=(-1)^n$

so $\frac{Q_n}{P_n}$ is convergent of the continued fraction expansion of $\sqrt{2}$ of the Pell equation $x^2-2y^2=1$.

**Proof:** From the proposition 1; two question arises

- Is $\sum_{n=1}^{\infty}\frac{1}{P_n^k}$ irrational for $k\geq 1$?
- Is $\sum_{n=1}^{\infty}\frac{1}{Q_n^k}$ irrational for $k\geq 1$?

**The behavior of $\zeta_q(2)$:** For $|q|<1$, defined q-derivative as:

$$\left(\frac{d}{d_qx}\right)f(x)=\frac{f(x)-f(qx)}{(1-q)x}$$

For $x\in\mathbb{R}$, defined q-integral as

$$\int_0^x f(t)d_qt=(1-q)\sum_{k=0}^{\infty}f(q^kx)q^kx$$

Note that

$$\int_0^x\left(\frac{d}{d_qx}\right)f(t)d_qt=f(x)$$

$$\int_0^{qx}f'(t)d_qt=f(qx)$$

and

$$q\int_0^1\int_0^1\frac{x^ry^r}{1-qxy}d_qxd_qy=\zeta_q(2)-\sum_{k=1}^{r}\frac{q^k}{[k]^2}$$

if $r=s$,

$$=\frac{q^{-s}}{r-s}\sum_{m=s+1}^{r}\frac{1}{[m]}$$

if $r\neq s$, where

$$r,s\in\mathbb{Z}_+,\ [x]=\frac{1-q^x}{1-q}$$

and

$$\zeta_q(s)=\frac{(2-s)(1-q)}{s-1}\sum_{n=1}^{\infty}\frac{q^n}{[n]^{s-1}}+\sum_{n=1}^{\infty}\frac{q^n}{[n]^s}$$

The q-binomial formula is

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}x^k,\ n\in Z_+$$

for

$$\binom{n}{k}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},(x;q)_n=(1-qx)\cdots\cdots(1-q^{n-1}x)$$

Thus

$$(q^{-n}x;q)_n=\sum_{k=0}^{n}\frac{(q^{-n};q)}{(q;q)_k}x^k$$

Let

$$P_{nq}(x)=\frac{1}{[n]!}\left(\frac{d}{d_qx}\right)^n\left(x^n(q^{-n+1}x;q)_n\right)$$

and

$$I_{nq}=q\int_0^1\int_0^1\frac{(1-y)^n}{1-qxy}P_{nq}(x)d_qxd_qy$$

It is easy to see that

$$\int_0^x f(t)\left(\frac{d}{d_qt}\right)g(t)d_qt=f(x)g(x)-\int_0^x f(t)g(qt)d_qt$$

$$\left(\frac{d}{d_qx}\right)^n\left(\frac{1}{1-qxy}\right)=n!y^nq^n\left(\frac{1}{\sum_{i=1}^{n+1}(1-xyq^i)}\right)=\frac{n!y^nq^n}{(xyq;q)_{n+1}}$$

$$\int_0^x\frac{P_{nq}(x)}{1-qxy}d_qx=(-1)^n n!y^nq^n\int_0^1\frac{(q^nx)^n(qx;q)_n}{(xyq;q)_{n+1}}d_qx$$

Finally we get

$$I_{nq}=q^{n^2+n+1}(-1)^n\int_0^1\int_0^1\frac{x^n(qx;q)_ny^n(1-y)^n}{(qxy;q)_{n+1}}d_qxd_qy$$

Let

$$d_n(q)=\ell cm\{1-q,1-q^2,\cdots\cdots,1-q^n\}$$

Then

$$d_n^2(q)q^{\frac{n^2-n}{2}}I_n(q)=d_n^2(q)A_n(q)\zeta_q(2)-B_n(q)$$

where

$$A_n(q),\ B_n(q) \in Z[q]$$

for

$$q = \frac{1}{b}\left(b(>2) \in Z\right)$$

$$A_n = b^{n^2+n+\frac{3n^2}{\pi^2}+\frac{n(n-1)}{2}} A_n(q) d_n^2(q)$$

$$B_n = b^{n^2+n+\frac{3}{\pi^2}n^2+\frac{n(n-1)}{2}} B_n(q)$$

As $A_n, B_n \in \mathbb{Z}$ then

$$\begin{aligned} \left|A_n\zeta(2) - B_n\right| &= d_n^2(q) q^{\frac{n^2-n}{2}} I_n(q) b^{n^2+n+\frac{3}{\pi^2}n^2+\frac{n(n-1)}{2}} \\ &\le M \cdot q^{\frac{n^2-n}{2}} b^{n^2+n+\frac{3}{\pi^2}n^2+\frac{n(n-1)}{2}} q^{\frac{n(n-1)}{2}+n^2n+1} \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

Hence $\zeta_q(2)$ is irrational.

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