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# On Certain Classes of Analytic Functions Defined by a Linear Operator

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**Abstract:** In this paper, we define and study two new classes of analytic functions which generalize a number of classes studied earlier. Sharp coefficient bound and some inclusion results are discussed. We also observe that these classes are preserved under the Bernadi integral transform.

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## INTRODUCTION

Let *A* be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

analytic in the open unit disk E = {z: |z|<1}. If *f*(z) and g(z) are analytic in *E,* we say that *f*(z) is subordinate to g(z), written $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function w(z) in *E* such that *f*(z) = g(w(z)).

In [1], Janowski introduced the class P[A,B]. For $-1 \le B < A \le 1$, a function p(z), analytic in E with p(0) = 1 belongs to the class P[A,B] if p(z) is subordinate to $\frac{1+Az}{1+Bz}$. Later Noor [2] generalize this concept to define the class $P_k[A,B]$ as follows.

An analytic function p(z) is said to be in the class $P_k[A,B]$, if and only if,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (1.2)$$

where $p_1(z)$, $p_2(z) \in P[A,B]$, $-1 \le B < A \le 1$, $k \ge 2$. It is clear that $P_2[A,B] \equiv P[A,B]$ and $P_k[1,-1] \equiv P_k$, the well-known class given and studied by Pinchuk [3].

By using all these concepts, we consider the following classes.

$$R_k[A,B] = \left\{ f(z) \in A : \frac{zf'(z)}{f(z)} \in P_k[A,B], z \in E \right\}$$

$$T_k[A,B] = \left\{ \begin{array}{l} f(z) \in A : \dfrac{zf'(z)}{g(z)} \in P_k[A,B], \\ g(z) \in R_2[1,-1], \ z \in E \end{array} \right\}$$

where $-1 \le B < A \le 1$, $k \ge 2$. These classes were studied by Noor [2, 4, 5].

For any two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \text{ and } g(z) = \sum_{n=0}^{\infty} a_n z^n \ (z \in E)$$

the convolution (Hadamard product) of *f*(z) and g(z) is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \ (z \in E)$$

Let H (a,b,c;z) be the hypergeometric function define as follows.

$$H(a,b,c;z) = 1 + \frac{ab}{c}\frac{z}{1} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2} + \cdots, z \in E \qquad (1.3)$$

where the value of a,b,c is not equal to 0, -1, -2,…,. We note that the series (1.3) converge absolutely for all $z \in E$ so that it represents an analytic function in *E*. For the applications of the hypergeometric type polynomial [6-15].

We consider the Integral operator $I_\lambda(a,b,c)$: $A \to A$ define by

$$I_\lambda(a,b,c) f(z) = \left(H(a,b,c;z)\right)^{-1} * f(z), \ z \in E \qquad (1.4)$$

where a,b,c are real and greater than zero and $(H(a,b,c;z))^{-1}$ is given by

$$H(a,b,c;z) * \left(H(a,b,c;z)\right)^{-1} = \frac{z}{(1-z)^{\lambda+1}}, \quad z \in E$$

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The operator $I_\lambda(a,b,c)$ was discussed by Noor [16] and known as generalized integral operator. In particular, with b = 1, this operator was studied in [17] for p-valent functions and for a = n+p, b = c, $\lambda = 1$, [18].
By some computation, we note that

$$I_\lambda(a,b,c)f(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad z \in E$$

With

$$A_n = \frac{(\lambda+1)_{n-1}(c)_{n-1}}{(a)_{n-1}(b)_{n-1}} a_n \tag{1.5}$$

where $(\rho)_n$ is a Pochhammar symbol given as

$$(\rho)_n = \begin{cases} 1, n = 0 \\ \rho(\rho+1)\dots(\rho+n-1), \ n \in \mathbb{N} \end{cases}$$

From (1.4), we note that

$$I_0(a,1,a)f(z) = f(z), \quad I_1(a,1,a)f(z) = zf'(z)$$

Also the following identities can easily be established.

$$(\lambda+1)I_{\lambda+1}(a,b,c)f(z) = \lambda I_\lambda(a,b,c)f(z) + z\big(I_\lambda(a,b,c)f(z)\big)' \tag{1.6}$$

$$aI_\lambda(a,b,c)f(z) = (a-1)I_\lambda(a+1,b,c)f(z) + z\big(I_\lambda(a+1,b,c)f(z)\big)' \tag{1.7}$$

We now define the following classes by using the operator $I_\lambda(a,b,c)$.

**Definition 1.1:** A function $f(z) \in A$ is in the class $R_k^\lambda(a,b,c,A,B)$, if and only if

$$\frac{z\big(I_\lambda(a,b,c)f(z)\big)'}{I_\lambda(a,b,c)f(z)} \in P_k[A,B], \quad z \in E \tag{1.8}$$

where a,b,c are greater than zero and $\lambda > -1$.

**Definition 1.2:** A function $f(z) \in A$ is in the class $T_k^\lambda(a,b,c,A,B)$, if and only if, there exists $g(z) \in R_2^\lambda(a,b,c,1,-1)$, such that

$$\frac{z\big(I_\lambda(a,b,c)f(z)\big)'}{I_\lambda(a,b,c)g(z)} \in P_k[A,B], \quad z \in E \tag{1.9}$$

where a,b,c are greater than zero and $\lambda > -1$.

By giving specific values to $k,\lambda,a,b,c,A,B$ in $R_k^\lambda(a,b,c,A,B)$ and $T_k^\lambda(a,b,c,A,B)$, we obtain many important subclasses studied by various authors in earlier papers [2,4,5,19,20].

From (1.8) and (1.9), we note that

$$f(z) \in R_k^\lambda(a,b,c,A,B) \Leftrightarrow I_\lambda(a,b,c) \in R_k[A,B]$$

and

$$f(z) \in T_k^\lambda(a,b,c,A,B) \Leftrightarrow I_\lambda(a,b,c) \in T_k[A,B]$$

Throughout in this article we assume that $-1 \le B < A \le 1$, $k \ge 2$, $\lambda > -1$ and a,b,c is not equal to 0, -1, -2,…, unless otherwise mentioned.

## PRELIMINARY LEMMA

In order to derive our main results, we need the following Lemma.

**Lemma 2.1:** [21]. Let h(z) be convex in the open unit disk E and let $Q: E \to \mathbb{C}$ with Re Q(z)>0. If p(z) is analytic in E, then

$$p(z) + Q(z)zp'(z) \prec h(z)$$

implies that

$$p(z) \prec h(z)$$

## MAIN RESULTS

**Theorem 3.1:** Let $f(z) \in R_k^\lambda(a,b,c,A,B)$ and *f*(z) is of the form (1.1). Then

$$|a_n| \le \frac{(a)_{n-1}(b)_{n-1}\left(\frac{k(A-B)}{2}\right)_{n-1}}{(c)_{n-1}(n-1)!(\lambda+1)_{n-1}}, \forall \ n \ge 2 \tag{3.1}$$

This result is sharp.

**Proof:** Let $f(z) \in R_k^\lambda(a,b,c,A,B)$ and set

$$\frac{z\big(I_\lambda(a,b,c)f(z)\big)'}{I_\lambda(a,b,c)f(z)} = p(z) \tag{3.2}$$

where p(z) is analytic in E with p(0) = 1. Let

$$p(z) = 1 + \sum_{n=1}^{\infty} b_n z^n, \ z \in E$$

Then (3.2) becomes

$$\left(z+\sum_{n=2}^{\infty} nA_n z^n\right)=\left(z+\sum_{n=2}^{\infty} A_n z^n\right)\left(1+\sum_{n=1}^{\infty} b_n z^n\right)$$

Equation coefficients of $z^n$, we have

$$(n-1)A_n=\sum_{j=1}^{n-1} A_{n-j} b_j$$

Using the coefficients estimates $|b_n| \le \frac{k(A-B)}{2}$ for the class $P_k[A,B]$, we have

$$2(n-1)|A_n| = k(A-B)\sum_{j=1}^{n-1} |A_{n-j}|$$

By using mathematical induction, we have

$$|A_n| \le \frac{\left(\frac{k(A-B)}{2}\right)_{n-1}}{(n-1)!}, \quad \forall\ n\ge 2$$

and hence by using (1.5), we obtain (3.1).

The equality occurs for the function $f_0(z)$ given by

$$\frac{z\left(I_\lambda(a,b,c)f_0(z)\right)'}{I_\lambda(a,b,c)f_0(z)}=\left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+Az}{1+Bz}-\left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-Az}{1-Bz}$$

**Theorem 3.2:** For $\lambda \ge 0$ and a,b,c greater than zero

$$R_k^{\lambda+1}(a,b,c,A,B) \subset R_k^{\lambda}(a,b,c,A,B) \subset R_k^{\lambda}(a+1,b,c,A,B)$$

**Proof:** Let $f(z) \in R_k^{\lambda}(a,b,c,A,B)$ and set

$$\begin{aligned}\frac{z\left(I_\lambda(a,b,c)f(z)\right)'}{I_\lambda(a,b,c)f(z)}&=p(z)\\ &=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\end{aligned} \tag{3.3}$$

where p(z) is analytic in $E$ with p(0) = 1. Then by simple computation together with the identity (1.6), we have

$$\frac{z\left(I_{\lambda+1}(a,b,c)f(z)\right)'}{I_\lambda(a,b,c)f(z)}=p(z)+\frac{zp'(z)}{p(z)+\lambda}$$

and since $f(z) \in R_k^{\lambda}(a,b,c,A,B)$ it follows that

$$p(z)+\frac{zp'(z)}{p(z)+\lambda} \in P_k[A,B] \tag{3.4}$$

Define a function

$$\phi_\lambda(z)=\frac{\lambda}{\lambda+1}\frac{z}{1-z}+\frac{1}{\lambda+1}\frac{z}{(1-z)^2}$$

Using the same convolution technique as used by Noor [22] of $\phi_\lambda(z)$ with (3.3), we have:

$$\begin{aligned}p(z)+\frac{zp'(z)}{p(z)+\lambda}&=\left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z)+\frac{zp_1'(z)}{p_1'(z)+\lambda}\right)\\ &-\left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z)+\frac{zp_2'(z)}{p_2'(z)+\lambda}\right)\end{aligned}$$

By using (3.4), we see that

$$\left\{p_i(z)+\frac{zp_i'(z)}{p_i(z)+\lambda}\right\} \in P[A,B]$$

for i = 1,2. Now by using (3.3) and Lemma 2.1, we obtain that

$$R_k^{\lambda+1}(a,b,c,A,B) \subset R_k^{\lambda}(a,b,c,A,B)$$

and by similar lines we have

$$R_k^{\lambda}(a,b,c,A,B) \subset R_k^{\lambda}(a+1,b,c,A,B)$$

For a function $f$(z) analytic in $E$, we consider the integral operator

$$F(z)=I_\gamma(f(z))=\frac{\gamma+1}{z^\gamma}\int_0^z t^\gamma f(t)dt,\ \gamma>-1 \tag{3.5}$$

The operator $I_\gamma(\gamma \in N)$, was introduced by Bernadi [23]. In particular, the operator $I_1$, was studied earlier by Libra [24] and Livingston [25].

**Theorem 3.3:** If $f(z) \in R_k^{\lambda}(a,b,c,A,B)$, then so does F(z), where F(z) is given by (3.5).

**Proof:** From (3.5), we have

$$\begin{aligned}(\gamma+1)I_\lambda(a,b,c)f(z)&=\gamma I_\lambda(a,b,c)f(z)\\ &+z\left(I_\lambda(a,b,c)f(z)\right)'\end{aligned} \tag{3.6}$$

Let

$$\frac{z\left(I_\lambda(a,b,c)F(z)\right)'}{I_\lambda(a,b,c)F(z)}=p(z)$$

where p(z) is analytic in E with p(0) = 1. Then by using (3.6), we have

$$\frac{z\left(I_{\lambda+1}(a,b,c)f(z)\right)'}{I_\lambda(a,b,c)f(z)} = p(z) + \frac{zp'(z)}{p(z)+\gamma}$$

and since $f(z)\in R_k^\lambda(a,b,c,A,B)$, it follows that

$$p(z)+\frac{zp'(z)}{p(z)+\gamma}\in P_k[A,B] \tag{3.7}$$

Let

$$\phi_\gamma(z) = \frac{\gamma}{\gamma+1}\frac{z}{1-z} + \frac{1}{\gamma+1}\frac{z}{(1-z)^2}$$

Then by convolution of $\phi_\gamma(z)$, we have

$$p(z)+\frac{zp'(z)}{p(z)+\gamma} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(p_1(z)+\frac{zp_1'(z)}{p_1'(z)+\gamma}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(p_2(z)+\frac{zp_2'(z)}{p_2'(z)+\gamma}\right)$$

Using (3.7), we see that

$$\left\{p_i(z)+\frac{zp_i'(z)}{p_i(z)+\gamma}\right\}\in P[A,B]$$

for i = 1,2. Now by using Lemma 2.1, we have $p_i(z)\in P[A,B]$ and hence $p(z)\in P_k[A,B]$. This completes the proof.

**Theorem 3.4:** Let $f(z)\in T_k^\lambda(a,b,c,A,B)$ and $f(z)$ is of the form (1.1). Then

$$|a_n| \le \frac{(a)_{n-1}(b)_{n-1}\left(k(A-B)(n-1)+4\right)}{4(c)_{n-1}\,(\lambda+1)_{n-1}},\ \forall\, n\ge 2$$

This result is sharp.

**Proof:** From the definition of the class $T_k^\lambda(a,b,c,A,B)$, it follows that there exist a function g(z) such that $I_\lambda(a,b,c)g(z)$ belongs to the class $S^*$ of starlike functions. Let us denote

$$I_\lambda(a,b,c)g(z) = z+\sum_{n=2}^{\infty}c_n z^n,\ z\in E$$

and

$$\frac{z\left(I_\lambda(a,b,c)f(z)\right)'}{I_\lambda(a,b,c)g(z)} = 1+\sum_{n=1}^{\infty}b_n z^n \tag{3.8}$$

Equating the coefficients of the power series in (3.8), we find from (1.5) that

$$\frac{n(\lambda+1)_{n-1}(c)_{n-1}}{(a)_{n-1}(b)_{n-1}}a_n = c_n + \sum_{j=1}^{n-1}c_{n-j}b_j \tag{3.9}$$

It is well known that the coefficients bounds for the class $S^*$ and $P_k[A,B]$ are $|c_n|\le n$, $|b_n|\le\frac{k(A-B)}{2}$ for all $n\ge 2$. Therefore (3.9) implies

$$\frac{n(\lambda+1)_{n-1}(c)_{n-1}}{(a)_{n-1}(b)_{n-1}}|a_n| \le n+\frac{k(A-B)}{2}\left[(n-1)+(n-2)+\cdots+2+1\right] = n+\frac{n(n-1)}{2}\frac{k(A-B)}{2},\ \forall\, n\ge 2$$

and thus we obtain the desired result.

The equality occurs for the function $f_0(z)$ given by

$$z\left(I_\lambda(a,b,c)f_0(z)\right)' = \left(I_\lambda(a,b,c)g_0(z)\right)p_k(z)$$

where

$$I_\lambda(a,b,c)g_0(z) = z+\sum_{n=2}^{\infty}nz^n$$

and

$$p_k(z) = 1+\sum_{n=1}^{\infty}\frac{1}{2}k(A-B)z^n$$

Using the same procedure as that of Theorem 3.2 and Theorem 3.3, we can easily prove the following results.

**Theorem 3.5:** For $\lambda\ge 0$ and a,b,c greater than zero

$$T_k^{\lambda+1}(a,b,c,A,B)\subset T_k^\lambda(a,b,c,A,B)\subset T_k^\lambda(a+1,b,c,A,B)$$

**Theorem 3.6:** If $f(z)\in T_k^\lambda(a,b,c,A,B)$, then the function $F(z)\in T_k^\lambda(a,b,c,A,B)$, where F(z) is given by (3.5).

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