World Applied Sciences Journal 12 (12): 2184-2188, 2011
ISSN 1818-4952

Analytical Solution of Nonlinear Thermoelasticity Cauchy Problem

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Abstract: In this paper, Homotopy Perturbation Method (HPM) is applied to solve the Cauchy problem arising in one dimensional nonlinear thermoelasticity. The comparison of the numerical solutions obtained by HPM with the exact solution shows the efficiency of this method.

Key words: Homotopy perturbation method . cauchy problem . nonlinear one dimensional thermoelasticity

INTRODUCTION

He [1-3] proposed a perturbation technique, namely Homotopy Perturbation Method (HPM) which does not require the small parameter assumtion and is coupled with all the positive features of homotopy and perturbation methods. The subsequent work [4-22] explicitly reveal the complete reliability of this powerful mathematical tool.The object of this study is to employ HPM to solve a real-life problem that exhibits coupling between the mechanical and thermal fields. Let us consider the following nonlinear system arising in thermoelasticity [23-25]:

$$u_{tt} - a(u_x,\theta)u_{xx} + b(u_x,\theta)\theta_x = f(x,t) \tag{1}$$

$$c(u_x,\theta)\theta_t + b(u_x,\theta)u_{xt} - d(\theta)\theta_{xx} = g(x,t) \tag{2}$$

subject to the initial conditions of

$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), \theta(x,0) = \theta^0(x) \tag{3}$$

where u (x,t) is the body displacement form equilibrium and θ(x,t) is the difference of the body's temperature from a reference $T_0 = 0$, subscripts denote partial derivatives and *a*, *b*, *c* and *d* are given smooth functions. For more details about the physical meaning of the model [23, 26]. Recently Ganji *et al*. [27] used Adomian decomposition method for solving the governing problem.

IMPLEMENTATION OF HPM TO THERMOELASTICITY PROBLEM

In order to illustrate the effectiveness of the method, an artificial model is used. Let us define *a*, *b*, *c*, *d*, u^0, u^1 and θ^0 by [25]:

$$a(u_x,\theta) = 2 - u_x\theta, b(u_x,\theta) = 2 + u_x\theta$$
$$c(u_x,\theta) = 1, d(u_x,\theta) = \theta \tag{4}$$

$$u^0(x) = \frac{1}{1+x^2}, u^1(x) = 0, \theta^0(x) = \frac{1}{1+x^2} \tag{5}$$

and replace the right-hand side of above equations by:

$$f(x,t) = \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w,v) - \frac{2x(1+t)}{(1+x^2)^2}b(w,v) \tag{6}$$

$$g(x,t) = \frac{2}{1+x^2}c(w,v) - \frac{4xt}{(1+x^2)^2}b(w,v) - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}d(v) \tag{7}$$

$$w \equiv w(x,t) = -\frac{2x(1+t^2)}{(1+x^2)^2}, v \equiv v(x,t) = \frac{1+t}{1+x^2} \tag{8}$$

where *a*, *b*, *c* and *d* are defined by Eq. (4) and the exact solution of two equations are given by [25]:

$$u(x,t) = \frac{1+t^2}{(1+x^2)^2}, \theta(x,t) = \frac{1+t}{1+x^2} \tag{9}$$

If we put (4) into (1) and (2), then we get:

$$u_{tt} - (2-u_x\theta)u_{xx} + (2+u_x\theta)\theta_x - f(x,t) = 0 \tag{10}$$

$$\theta_t + (2+u_x\theta)u_{xt} - \theta\theta_{xx} - g(x,t) = 0 \tag{11}$$

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and we get

$$u_{tt} - 2u_{xx} + u_x u_{xx}\theta + 2\theta_x + u_x \theta_x \theta - f(x,t) = 0 \tag{12}$$

$$\theta_t + 2u_{xt} + u_x u_{xt}\theta - \theta\theta_{xx} - g(x,t) = 0 \tag{13}$$

We construct the following homotopies

$$u_{tt} + p\{\ -2u_{xx} + u_x u_{xx}\theta + 2\theta_x + u_x\theta_x\theta - f(x,t)\ \} = 0 \tag{14}$$

$$\theta_t + p\{\ 2u_{xt} + u_x u_{xt}\theta - \theta\theta_{xx} - g(x,t)\ \} = 0 \tag{15}$$

Assume the solution of Eqs. (14,15) to be in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \dots \tag{16}$$

$$\theta = \theta_0 + p\theta_1 + p^2\theta_2 + p^3\theta_3 + \dots \tag{17}$$

Substituting (16-17) into (14,15) and equating the coefficients of like powers p, we get the following set of differential equations

$$p^0 : (u_0)_{tt} = 0$$
$$(\theta_0)_t = 0$$

$$p^1 : (u_1)_{tt} - 2(u_0)_{xx} + (u_0)_x (u_0)_{xx}\theta_0 + 2(\theta_0)_x + (u_0)_x(\theta_0)_x\theta_0 - f(x,t) = 0$$

$$(\theta_1)_t + 2(u_0)_{xt} + (u_0)_x(u_0)_{xt}\theta_0 - \theta_0(\theta_0)_{xx} - g(x,t) = 0$$

$$p^2 : (u_2)_t - 2(u_1)_{xx} + (u_0)_x(u_0)_{xx}\theta_1 + (u_0)_x(u_1)_{xx}\theta_0 + (u_1)_x(u_0)_{xx}\theta_0 + 2(\theta_1)_x + (u_0)_x(\theta_1)_x\theta_0 + (u_0)_x(\theta_0)_x\theta_1 + (u_1)_x(\theta_0)_x\theta_0 - f(x,t) = 0$$

$$(\theta_2)_t + 2(u_1)_{xt} + (u_0)_x(u_0)_{xt}\theta_1 + (u_0)_x(u_1)_{xt}\theta_0 + (u_1)_x(u_0)_{xt}\theta_0 - \theta_0(\theta_1)_{xx} - \theta_1(\theta_0)_{xx} - g(x,t) = 0$$

$$p^3 : (u_3)_t - 2(u_2)_{xx} + (u_1)_x(u_1)_{xx}\theta_0 + (u_0)_x(u_0)_{xx}\theta_2 + (u_0)_x(u_1)_{xx}\theta_1 + (u_1)_x(u_0)_{xx}\theta_1 + (u_2)_x(u_0)_{xx}\theta_0 - f(x,t) = 0$$

$$(\theta_3)_t + 2(u_2)_{xt} + (u_0)_x(u_0)_{xt}\theta_2 + (u_0)_x(u_1)_{xt}\theta_1 + (u_0)_x(u_2)_{xt}\theta_0 + (u_1)_x(u_1)_{xt}\theta_1 + (u_1)_x(u_1)_{xt}\theta_0 + (u_2)_x(u_0)_{xt}\theta_0 - \theta_0(\theta_2)_{xx} - \theta_2(\theta_0)_{xx} - \theta_1(\theta_1)_{xx} - g(x,t) = 0$$

$\dots$

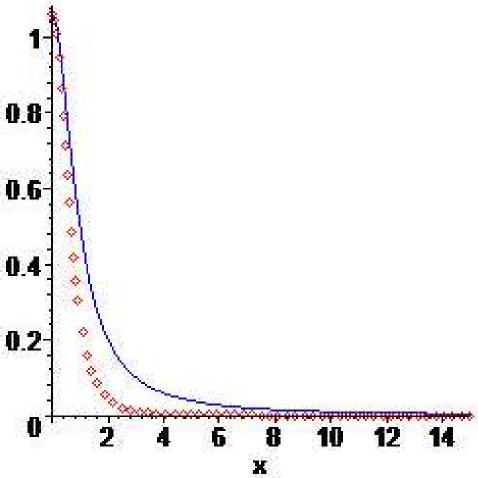

Fig. 1: u(x,t) when t = 0.25 Line: HPM, Point: exact

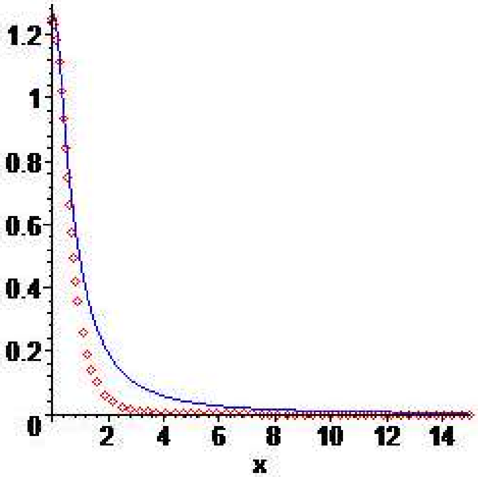

Fig. 2: u(x,t) when t = 0.25 Line: HPM, Point: exact

and so on, the rest of the polynomials can be constructed in a similar manner. With the initial conditions Eq. (3) gives

$$u_0(x,t) = \frac{1}{1+x^2} \tag{18}$$

$$u_1(x,t) = \frac{1}{105(1+x^2)^6}\begin{pmatrix} 10t^7(x-3x^3) + 14t^6(x+x^2-3x^3+x^4) \\ +42t^5(x+x^2-3x^3+x^4) \\ +35t^4(1+x+2x^2-6x^3-2x^4-8x^6-3x^8) \\ +70t^3(2x^2-7x^3+2x^4-6x^5-4x^7-x^9) \\ +105t^2(1+5x^2+10x^4+10x^6+5x^8+x^{10}) \end{pmatrix} \tag{19}$$

$$\theta_0(x,t) = \frac{1}{1+x^2} \tag{20}$$

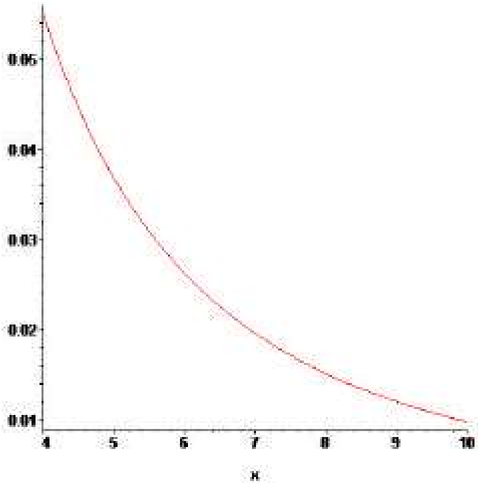

Fig. 3: Absolute error when t = 0.25

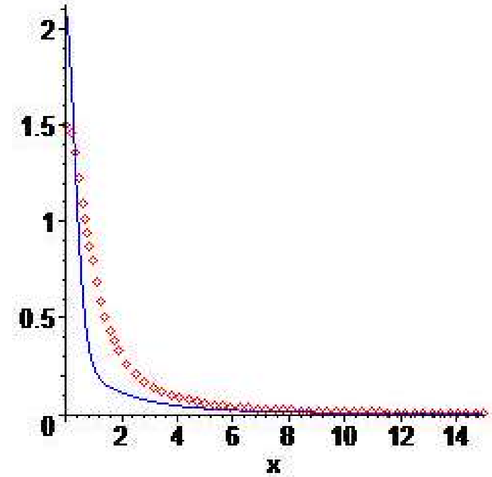

Fig. 6: θ(x,t) when t = 0.5 Line: HPM, Point: exact

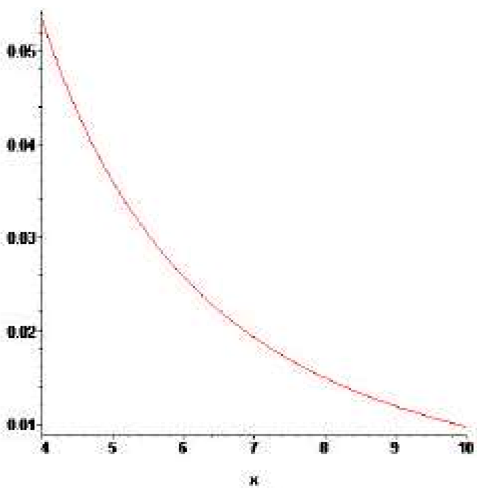

Fig. 4: Absolute error when t = 0.5

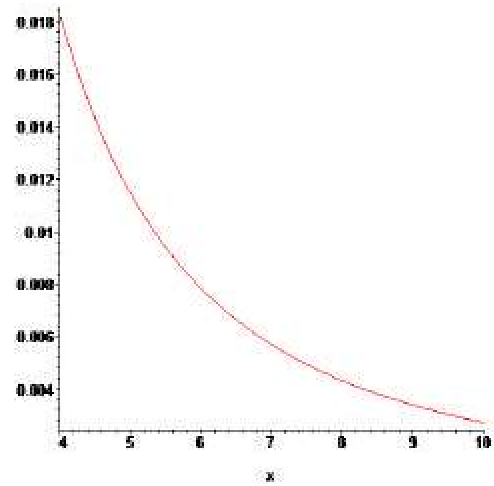

Fig. 7: Absolute error when t = 0.25

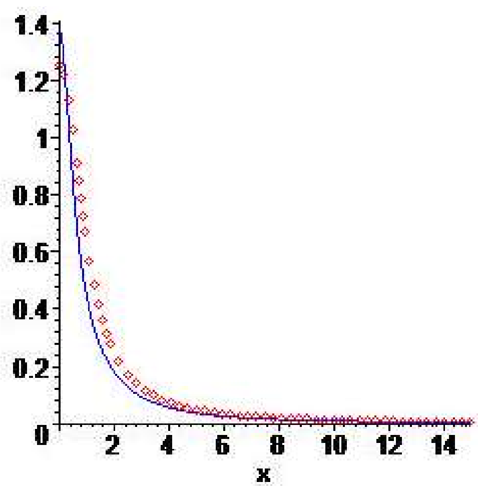

Fig. 5: θ(x,t) when t = 0.25 Line: HPM, Point: exact

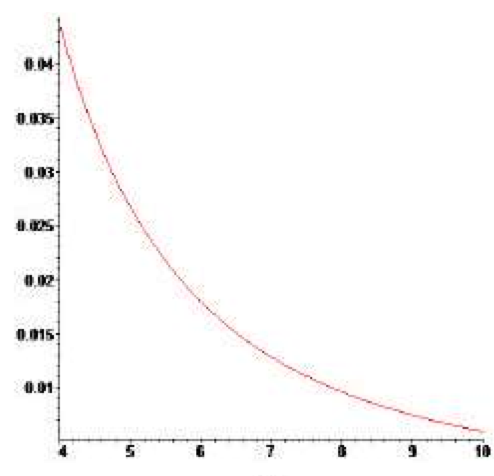

Fig. 8: Absolute error when t = 0.5

$$\theta_1(x,t)=\frac{1}{15\left(1+x^2\right)^5}\begin{pmatrix}24t^5x^2+30t^4x^2+10t^3\left(1+2x^2-3x^4\right)\\ +30t^2\left(1-2x-6x^3-3x^4-6x^5-2x^7\right)\\ +15t\left(1+4x^2+6x^4+4x^6+x^8\right)\end{pmatrix} \quad (21)$$

Proceeding in the same way, we can obtain $u_2(x,t)$, $\theta_2(x,t)$ and higher order approximations. Here, the numerical results are evaluated using terms approximation of the recursive relations.

CONCLUSIONS

In this study, we have successfully applied HPM to obtain an approximation of the analytic solution of the Cauchy problem arising in one dimensional nonlinear thermoelasticity. In this method, the solution is found in the form of a convergent series with easily computed components. The results obtained by homotopy perturbation method are compared with those of the exact solution, which shows very good agreement, even using only few terms of the recursive relations. In general, this method provides highly accurate numerical solutions and can be applied to wide class of nonlinear problems. Homotopy perturbation method does not require small parameters which are needed by perturbation method. Also the method avoids linearization and physically unrealistic assumptions.

ACKNOWLEDGMENT

The first author is highly grateful to Brig (Rtd) Qamar Zaman Vive Chancellor HITEC University Pakistan for the provision of excellent research facilities coupled with the appropriate conducive environment.

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