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Varietal Nilpotent Groups

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Abstract: Let ϑ be a variety of groups defined by a set V of laws. A group ϑ is said to be ϑ-nilpotent if there exists series normal of ϑ where quotient groups contained in ϑ-marginal factor of ϑ. In this note, it is shown that if ϑ be a ϑ-nilpotent group and $N \trianglelefteq G$ such that $|N| = p^n$ then N contained in $V^*n(G)$.

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Key words: Variety . verbal . marginal . nilpotent

INTRODUCTION

Let F be a free group on a count ably infinite set $\{x_1, x_2, \ldots\}$ and let V be a subset of F and ϑ be a variety of groups defined by a set V of laws [3].

Let ϑ be an arbitrary group with an normal subgroup N, then we define the verbal subgroup V(G) and the marginal subgroup $V^*(G)$ and $[NV^*G]$ as follows

$$V(G) = \{v\,(g_1, g_2, \ldots, g_n); v \in V\ , g_i \in G\ , 1 \le i \le n\}$$

$$V^*(G) = \left\{ \begin{matrix} g \in G\ ; v(g_1, g_2, \ldots, g_i g, \ldots, g_n) = v(g_1, g_2, \ldots, g_i, \ldots, g_n)\ , \\ v \in V\ , g_i \in G\ , 1 \le i \le n \end{matrix} \right\}$$

$$\left[NV^*G\right] = \left\{ \begin{matrix} v(g_1, g_2, \ldots, g_i a, \ldots, g_n)\ v(g_1, g_2, \ldots, g_i, \ldots, g_n)^{-1}; \\ v \in V\ , g_i \in G, a \in N, 1 \le i \le n \end{matrix} \right\}$$

It is easy to checked that the verbal subgroup V(G) is a fully invariant subgroup and the marginal subgroup $V^*(G)$ is a characteristic subgroup in ϑ and $[NV^*G]$ is the smallest normal subgroup of G contained in N, such that $\frac{N}{\left[NV^*G\right]} \le V^*\left(\frac{G}{\left[NV^*G\right]}\right)$.

The following lemma give basic properties of verbal and marginal subgroups of a group G with respect to the variety, which are useful in our investigation [1, 2] for the proofs.

Lemma 1.1: Let ϑ be a variety of groups defined by the set of laws and let N be a normal subgroup of a group ϑ. Then the following statements hold:

(*i*) $V\left(V^*(G)\right) = 1$ *and* $V^*\left(\frac{G}{V(G)}\right) = \frac{G}{V(G)}$

$(ii)\ V(G) = 1$ *iff* $V^*(G) = G$ *iff* $G \in \vartheta$;

$(iii)\ \left[NV^*G\right] = 1$ *iff* $N \le V^*(G)$;

$(iv)\ V\left(\frac{G}{N}\right) = \frac{V(G)N}{N}$ and $\frac{V^*(G)N}{N} \le V^*\left(\frac{G}{N}\right)$;

$(v)\ V(N) \le \left[NV^*G\right] \le N \cap V(G)$.inparticular, $V(G) = \left[GV^*G\right]$;

(vi) $N \cap V(G) = 1$ *then* $N \le V^*(G)$ *and* $V^*(G/N) = (V^*(G))/N$

(vii) $[G, N] \le V^*(G)$ *and* $\left[V(G), V^*(G)\right] = 1$

(viii)(G)$\cap V^*(G)$, is contained in the Frattini subgroup of G.

Theorem 1.2: [1] Let $H \le G$ and $N \lhd G$ such that G = HN. Let ϑ be a variety. Then V(G) = V(H) $[NV^*G]$

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Let ϑ be a variety of groups, we define the lower ϑ-verbal series of ϑ to be

$$G = V^0(G) \geq V^1(G) \geq \ldots \geq V^n(G) \geq \ldots \quad ,$$

where, for n>0, $V^n(G) = V(V^{n-1}(G))$. It is easy that

$$\frac{V^{n-1}(G)}{V^n(G_0)}$$

The upper θ-marginal series of G to be

$$1 = V_0^*(G) \leq V_1^*(G) \leq \ldots \leq V_n^*(G) \leq \ldots ,$$

where, for $n > 0$, $\dfrac{V^*_n(G)}{V^*_{n-1}(G)} = V^*\left(\dfrac{G}{V^*_{n-1}(G)}\right)$.

The corresponding lower θ-marginal series of G given by

$$G = V_0(G) \geq V_1(G) \geq \ldots \geq V_n(G) \geq \ldots,$$

where, for n>0, $V_n(G) = [V_{n-1}(G)V^*G]$ also [4,5].
By using definition and lemma 1.1, the following properties hold, for i,j≥0

(i) $V^i(V^j(G)) = V^{i+j}(G)$;

(ii) $V^*_i\left(\dfrac{G}{V^*_j(G)}\right) = \dfrac{V^*_{i+j}(G)}{V^*_j(G)}$;

(iii) $\dfrac{V_i(G)}{V_{i+1}(G)} \leq V^*\left(\dfrac{G}{V_{i+1}(G)}\right)$;

(iv) $V^*\left(\dfrac{V^i(G)}{V^{i+1}(G)}\right) = \dfrac{V^i(G)}{V^{i+1}(G)}$;

Difinition 1.3: A group G is said to be ϑ-nilpotent if there exist a series

$$1 = G_0 \leq G_1 \leq \ldots \leq G_k = G$$

Such that $G_i \lhd G$ and $\dfrac{G_{i+1}}{G_1} \leq V^*(\dfrac{G}{G_1})$, for i = 0, 1,…,n-1.

The length of the shortest series (*) is the ϑ-nilpotent class of G.

The class of ϑ-nilpotent groups is closed under the formation of subgroups, images and finite direct products.

Theorem 1.4: [4] A group G is a ϑ-nilpotent of class n iff

$$V_{n+1}(G) = 1 \text{ iff } V^*n(G) = G$$

Thworem 1.5: [5] If V = {[$x_1, x_2, \ldots, x_n$]} and ϑ be a variety of groups defined by V and G be an arbitrary group such that $\dfrac{G}{V^*G_0}$ is a cyclic group then $V^*(G) = G$.

Theorem 1.6: [4] If G is a ϑ-nilpotent group and $1 \neq N \trianglelefteq G$, then $N \cap V^*(G) \neq 1$.

MAIN RESULT

Let G be an arbitrary group and ϑ be a variety of groups. If $V^*(G)$ is trivial then one easily shows that $V^*_n(G) = 1$, for n≥1. In this case if G is a ϑ-nilpotent group then G is trivial group.

Theorem 2.1: If V = {[$x_1, x_2, \ldots, x_n$]} and ϑ be a variety of groups defined by V and G be a finite P-group where P is prime number then G is a ϑ-nilpotent group.

Proof: Let $|G| = p^m$ by induction on the order of G. If |G| = P and if $|V^*(G)| = 1$ implies that $\left|\dfrac{G}{V^*(G)}\right| = P$ by theorem 1.5, i.e. $|V^*(G)| = P$ and this is contradiction with assumption, hence $|V^*(G)| = P$ i.e. G is a ϑ-nilpotent group of class one. Now assume that m≥1 and $|G| = p^m$ if $V^*(G) = G$ then G is ϑ-nilpotent group otherwise the oder of $\dfrac{G}{V^*(G)}$ is less than p^m by the induction hypothesis, this group is a ϑ-nilpotent group and has the upper ϑ-marginal series as follows that

$$1 = \frac{G_0}{V^*(G)} \leq \frac{G_1}{V^*(G)} \leq \ldots \leq \frac{G_k}{V^*(G)} = \frac{G}{V^*(G)}$$

by isomorphic theorems we have $\dfrac{G_i}{G_{i-1}} \leq V^*\left(\dfrac{G}{G_{i-1}}\right)$ it follows that there exist a upper ϑ-marginal series for G as follows that

$$1 = G_0 \leq G_1 \leq \ldots \leq G_k = G$$

i.e. G is a ϑ-nilpotent group. ■

Corollary 2.2: If V = {[$x_1, x_2, \ldots, x_n$]} and ϑ be a variety of groups defined by V and G be a ϑ-nilpotent group of class C>1 then $V^*_{c-1}(G)$ is not cyclic.

Proof: let $\dfrac{G}{V^*_{(c-1)}(G)}$ be cyclic, then

$$\frac{\dfrac{G}{V^*_{c-2}(G)}}{V^*\left(\dfrac{G}{V^*_{c-2}(G)}\right)} = \frac{\dfrac{G}{V^*_{c-2}(G)}}{\dfrac{V^*_{c-1}(G)}{V^*_{c-2}(G)}} \simeq \frac{G}{V^*_{c-1}(G)}$$

By theorem 1.5 we have

$$\frac{V^*_{c-1}(G)}{V^*_{c-2}(G)} = V^*\left(\frac{G}{V^*_{c-2}(G)}\right) = \frac{G}{V^*_{c-2}(G)}$$

i.e. G = $V^*_{c-1}(G)$ and this is contradiction with assumption.

Theorem 2.3: If G be a group, $N \le V^*(G)$ and G/N be a ϑ-nilpotent group then G is a ϑ-nilpotent group.

Proof: Let G/N be a ϑ-nilpotent group thus there exist a normal series as follows

$$1 = \frac{G_1}{N} \le \frac{G_2}{N} \le \ldots \le \frac{G_n}{N} = \frac{G}{N}$$

Such that

$$\frac{\dfrac{G_{i+1}}{N}}{\dfrac{G_i}{N}} \le V^*\left(\frac{\dfrac{G}{N}}{\dfrac{G_i}{N}}\right)$$

By isomorphic theorems we have $\dfrac{G_{i+1}}{G_i} \le V^*\left(\dfrac{G}{G_i}\right)$ and the normal series

$$1 = G_0 \le G_1 = N \le \ldots \le G_n = G$$

Such that $\dfrac{G_1}{G_0} \le V^*\left(\dfrac{G}{G_0}\right)$ thus G is a ϑ-nilpotent group.

Theorem 2.4: If G be a ϑ-nilpotent group and $N \trianglelefteq G$ such that $|N| = p^n$ then $N \le V^*_n(G)$.

Proof: By Induction we have if n = 1 and |N| = P then $1 \ne N \cap V^*(G) \le N$ thus $|N \cap V^*(G)| = P = |N|$ it follows that $N \cap V^*(G) = N$ hence $N \le V^*(G)$. Let the assertion hold for for every number less than n and $|N| = p^n$ and $M = N \cap V^*(G) \ne 1$ then $|N/M| = p^m$ and m<n. By isomorfhic theorems we have

$$\frac{N}{M} = \frac{N}{N \cap V^*(G)} \cong \frac{NV^*(G)}{V^*(G)}$$

By Induction hypothesis

$$\frac{NV^*(G)}{V^*(G)} \le V^*_m\left(\frac{G}{V^*(G)}\right) = \frac{V^*_{m+1}(G)}{V^*(G)}$$

that is implies that $N \le V^*_{m+1}(G) \le V^*_n(G)$

Theorem 2.5: Let ϑ be a variety of groups, G be a ϑ-nilpotent group of class n and G = H V(G) then G = H.

Proof: By Inductin we have G = H $V_n(G)$ hence by theorem 1.2 V(G) = V(H)[$V_n(G)V^*G$] it follows that G = H V(G) = H V(H)$V_{n+1}(G)$ = H $V_{n+1}(G)$ and since G is a ϑ-nilpotent group thus $V_{n+1}(G) = 1$ hence the result follows immediately.

Theorem 2.6: Let ϑ be a variety of groups and G be a finite group. Then there is a subgroup H such that G=H $V^*(G)$ and $H \cap V^*(G)$ is a ϑ-nilpotent group.

Proof: by Induction on the |G|, if $V^*(G) \subseteq \Phi(G)$ then trivially G= H and since $V(V^*(G)) = 1$ thus $V^*(G)$, is a ϑ-nilpotent. Thus $H \cap V^*(G) = V^*(G)$ the result follows. Now if $V^*(G) \not\subseteq \Phi(G)$ then there is a maximal subgroup M such that $V^*(G) \not\subseteq$ M. By hypothesis induction there is $H \le M$ such that M = H ($V^*(G) \cap M$) and $H \cap (V^*(G) \cap M) = H \cap V^*(G)$ is a ϑ-nilpotent group thus G = H $V^*(G)$ and the assertion hold.

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