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A New Analytical Approach to Solve Thomas-Fermi Equation

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Abstract: The aim of this paper is to introduce a new approximate method, namely the Modified Laplace Padé Decomposition Method (MLPDM) which is a combination of modified Laplace decomposition and Padé approximation to provide an analytical approximate solution to Thomas-Fermi equation. This new iteration approach provides us with a convenient way to approximate solution. A good agreement between the obtained solution and some well-known results has been demonstrated. The proposed technique can be easily applied to handle other strongly nonlinear problems.

Key words: Modified laplace decomposition method . padé approximants . thomas-fermi equation . approximate solution

INTRODUCTION

The Thomas-Fermi equation arises in the mathematical modeling of various models in physics, astrophysics, solid state physics, nuclear charge in heavy atoms and applied sciences [1-5]. Due to its diversified significance and importance, the Thomas-Fermi equation which is a second order nonlinear differential equation has been investigated by many researchers. Various powerful mathematical techniques such as Adomian decomposition method [6], variational iterative method [7] and differential transform method [8] have been proposed for obtaining the approximate analytic solution of the Thomas-Fermi equation.

The Laplace decomposition method was first proposed by Khuri [9, 10], with coupling of standard Adomian decomposition method and Laplace transform for solving nonlinear differential equations and Bratu's problem. There is no need of linearization, discretization and large computational work. It has been used to solve effectively, easily and accurately a large class of nonlinear problems with approximation. Recently Majid *et al*. have been introduced various modifications in Laplace decomposition to deal with nonlinear behaviors of the physical models [11-14]. It is worth mentioning that the proposed method is an elegant combination of the modified Laplace decomposition method and the Padé approximantions [15]. The advantage of this proposed method is its capability of combining two powerful methods for obtaining rapid convergent series for nonlinear equations. To the best of authors knowledge no attempt has been made to exploit this method to solve nonlinear Thomas-Fermi equation.

This paper considers the effectiveness of the modified Laplace Padé decomposition method for solving second order nonlinear Thomas-Fermi equation. The paper is organized as follows. In Section 2, the basic concept of MLPDM is presented. Section 3 contains basic idea of Padé approximants. Section 4, contains governing equations. The conclusions are given in last Section.

FORMULATION OF MODIFIED LAPLACE DECOMPOSITION METHOD

Consider equation F(u(x)) = g(x) where F represents a general nonlinear ordinary or partial differential operator including both linear and nonlinear terms. The linear terms are decomposes into L+R, where L is a linear operator and R is the remaining of the linear operator. Thus, the equation can be written as

$$Lu + Ru + Nu = g(x) \tag{1}$$

where Nu, indicates the nonlinear terms. By applying Laplace transform on both sides of Eq. (1), we get

$$\mathbf{L}\left[Lu + Ru + Nu = g(x)\right] \tag{2}$$

Using the differential property of Laplace transform, we have

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$$s^n \mathbf{L}[u] - \sum_{k=1}^{n} s^{k-1} u^{(n-k)}(0) + \mathbf{L}[Ru] + \mathbf{L}[Nu] = \mathbf{L}[g(x)] \quad (3)$$

Operating inverse Laplace transform on both sides of Eq. (3), we get

$$u = G(x) - \mathbf{L}^{-1}\left[\frac{1}{s^n}\left[\mathbf{L}[Nu] + \mathbf{L}[Ru]\right]\right] \quad (4)$$

The Laplace decomposition method assumes the solution u can be expanded into infinite series as

$$u = \sum_{m=0}^{\infty} u_m \quad (5)$$

Also the nonlinear term Nu can be written as

$$Nu = \sum_{m=0}^{\infty} A_m \quad (6)$$

where A_m are the Adomian polynomials [6]. By substituting Eqs. (5) and (6) in Eq. (4), the solution can be written as

$$\sum_{m=0}^{\infty} u_m(x) = G(x) - \mathbf{L}^{-1}\left[\frac{1}{s^n}\left[\mathbf{L}\left[\sum_{m=0}^{\infty} A_m\right] + \mathbf{L}\left[R\sum_{m=0}^{\infty} u\right]\right]\right] \quad (7)$$

In Eq. (7), the Adomian polynomials can be generated by several means. Here we used the following recursive formulation:

$$A_m = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, \quad m = 0,1,2,.... \quad (8)$$

In general, the recursive relation is given by

$$\begin{aligned} &u_0(x) = G(x) \\ &u_{m+1}(x) = -\mathbf{L}^{-1}\left[\frac{1}{s^n}\left[\mathbf{L}\left[\sum_{m=0}^{\infty} A_m\right] + \mathbf{L}\left[R\sum_{m=0}^{\infty} u\right]\right]\right], \quad m \ge 0 \end{aligned} \quad (9)$$

where G(x) represents the term arising from source term and prescribe initial conditions. The modified Laplace decomposition method [14] suggests that the function G(x) in Eqs. (10) can be decompose into two parts

$$G(x) = G_0(x) + G_1(x) \quad (10)$$

where $G_0(x)$ is assign to zeroth order solution and the remaining part $G_1(x)$ is assign to first order solution. Using that assumption we reformulate Eqs. (9) for modified Laplace decomposition method as:

$$\begin{cases} u_0(x) = G_0(x) \\ u_1(x) = G_1(x) - \mathbf{L}^{-1}\left[\frac{1}{s^n}\left[\mathbf{L}\left[\sum_{m=0}^{\infty} A_0\right] + \mathbf{L}\left[R\sum_{m=0}^{\infty} u_0\right]\right]\right] \\ u_{m+1}(x) = -\mathbf{L}^{-1}\left[\frac{1}{s^n}\left[\mathbf{L}\left[\sum_{m=0}^{\infty} A_m\right] + \mathbf{L}\left[R\sum_{m=0}^{\infty} u_m\right]\right]\right], \quad m \ge 1 \end{cases} \quad (11)$$

The proposed method does not resort to linearization or assumptions of weak nonlinearity, the solution generated in the form of general solution and it is more realistic compared to the method of simplifying the physical problems.

PADÉ APPROXIMANTS

A Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function u(x). The [L/M] Padé approximants to a function u(x) are given by Baker [5].

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)} \quad (12)$$

where $P_L(x)$ is a polynomial of degree at most L and $Q_M(x)$ is a degree of at most M. The power series in terms of x is given below

$$u(x) = \sum_{i=0}^{\infty} a_i x^i \quad (13)$$

$$u(x) = \frac{P_L(x)}{Q_M(x)} + O(x^{L+M+1}) \quad (14)$$

Determine the coefficients of $P_L(x)$ and $Q_M(x)$ by Eq. (14). We can multiply the numerator and denominator by a constant and leave [L/M] unchanged, we imposed the normalization condition

$$Q_M(x) = 1 \quad (15)$$

Expanding polynomials $P_L(x)$ and $Q_M(x)$ in power series in terms of x of order L and M which is given below:

$$\begin{aligned} P_L(x) &= p_0 + p_1 x + p_2 x^2 + ... + p_L x^L \\ Q_M(x) &= 1 + q_1 x + q_2 x^2 + ... + q_M x^M \end{aligned} \quad (16)$$

Using Eq. (16) in Eq. (14), we can write Eq. (14) in the notation of formal power series as

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + p_2 x^2 + ... + p_L x^L}{1 + q_1 x + q_2 x^2 + ... + q_M x^M} + O(x^{L+M+1}) \quad (17)$$

By cross-multiplication of Eq. (17), we get

$$\begin{aligned}&(p_0 + p_1 x + ... + p_L x^L)(a_0 + a_1 x + a_2 x^2 + ...)\\&= 1 + q_1 x + ... + q_M x^M + O(x^{L+M+1})\end{aligned} \tag{18}$$

From Eq. (18) we obtain the set of linear equations

$$\begin{cases} a_0 = p_0 \\ a_1 + a_0 q_1 = p_1 \\ a_2 + a_1 q_1 + a_0 q_2 = p_2 \\ \vdots \\ a_L + a_{L-1} q_1 + a_0 q_L = p_L \end{cases} \tag{19}$$

and

$$\begin{cases} a_{L+1} + a_L q_1 + ... + a_{L-M+1} q_M = 0 \\ a_{L+2} + a_{L+1} q_1 + ... + a_{L-M+2} q_M = 0 \\ \vdots \\ a_{L+M} + a_{L+M-1} q_1 + a_L q_M = 0 \end{cases} \tag{20}$$

From Eq. (20), we can obtain q_i, $1 \le i \le M$. Once the values of $q_1, q_2, \ldots, q_M$ are all known Eq. (19) gives an explicit formula for the unknown quantities $p_1, p_2, \ldots, p_L$ We calculate diagonal approximants like [2/2], [3/3], [4/4] or [5/5] which are more accurate than non diagonal approximants and can be calculated easily by built-in utilities of Mathematica 7 and Maple 14.

NUMERICAL APPLICATIONS

In this section, we apply new version of ADM for finding an approximate solution of the Thomas-Fermi equation. We also introduce a slight modification in the selection of initial value which makes the application of the proposed algorithms simpler and improves the efficiency. Consider the Thomas-Fermi equation [1].

$$f'' = \frac{f^{3/2}}{\eta} \tag{21}$$

$$f(0) = 1, f'(0) = 0 \tag{22}$$

where prime denote differentiation with respect to η. To apply modified Laplace decomposition method, we write Eq. (21) in an operator form

$$\frac{d^2 f}{d\eta^2} = \frac{f^{3/2}}{\eta} \tag{23}$$

Applying Laplace transform algorithm we get

$$s^2 \mathbf{L}[f] - sf(0) - f'(0) = \mathbf{L}\left[\eta^{-1/2} f^{3/2}\right] \tag{24}$$

Using given boundary conditions in Eq. (22) into Eq. (24), we have

$$s^2 \mathbf{L}[f] - s - \alpha = \mathbf{L}[\eta^{-1/2} f^{3/2}]$$

$$\mathbf{L}[f] = \frac{s+\alpha}{s^2} + \frac{1}{s^2}\mathbf{L}\left[\eta^{-1/2} f^{3/2}\right] \tag{25}$$

Applying inverse Laplace transform to Eq. (25) we get

$$f(\eta) = 1 + \alpha\eta + \mathbf{L}^{-1}\left[\frac{1}{s^2}\mathbf{L}\left[\eta^{-1/2} f^{3/2}\right]\right] \tag{26}$$

The Laplace decomposition method assumes a series solution of the function $f(\eta)$ is given by

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) \tag{27}$$

Substituting Eq. (27) into Eq. (26), we get

$$\sum_{m=0}^{\infty} f_m(\eta) = 1 + \alpha\eta + \mathbf{L}^{-1}\left[\frac{1}{s^2}\mathbf{L}\left[\eta^{-1/2}\sum_{m=0}^{\infty} A_m(\eta)\right]\right] \tag{28}$$

In Eq. (28), $A_m(\eta)$ is a Adomian polynomial that represents nonlinear term. By modified Laplace decomposition method, our modified recursive relation is given below

$$\begin{aligned} &f_0(\eta) = 1 \\ &f_1(\eta) = \alpha\eta + \mathbf{L}^{-1}\left[\frac{1}{s^2}\mathbf{L}\left[\eta^{-1/2} A_0(\eta)\right]\right] \\ &f_{m+1}(\eta) = \mathbf{L}^{-1}\left[\frac{1}{s^2}\mathbf{L}\left[\eta^{-1/2}\sum_{m=0}^{\infty} A_m(\eta)\right]\right],\ m \ge 1 \end{aligned} \tag{29}$$

Now the components of the series solution are

$$f_0(\eta) = 1 \tag{30}$$

$$f_1(\eta) = \alpha\eta + \frac{4\eta^{3/2}}{3} \tag{31}$$

$$f_2(\eta) = \frac{\eta^3}{3} + \frac{2\alpha\eta^{5/2}}{5} \tag{32}$$

$$f_3(\eta) = \frac{2\alpha\eta^4}{15} + \frac{2\eta^{9/2}}{27} + \frac{3\alpha^2\eta^{7/2}}{70} \tag{33}$$

$$\vdots$$

$$\vdots$$

The series solution is given by

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) \quad (34)$$

Substituting Eqs. (30)-(33), into Eq. (34), we obtain the following series solution

$$f(\eta) = 1 + \alpha\eta + \frac{4\eta^{3/2}}{3} + \frac{\eta^3}{3} + \frac{2\alpha\eta^{5/2}}{5} + \frac{2\alpha\eta^4}{15} + \frac{2\eta^{9/2}}{27} + \frac{3\alpha^2\eta^{7/2}}{70} + ... \quad (35)$$

Our aim in this section is mainly concerned with the mathematical behavior of the solution $f(\eta)$ in order to determine the value of free parameter $\alpha = f'(0)$. It was formally shown by that this goal can easily be achieved by forming the Padé approximants [15] which have the advantage of manipulating the polynomial approximation into a rational function to obtain the more information about $f(\eta)$. It is well known fact that Padé approximants will converges on the entire real axis if $f(\eta)$ is free of singularities on the entire real axis. More importantly, the diagonal approximants are most accurate approximants; therefore we will construct only diagonal approximants. Using the boundary condition $f(\infty) = 0$, the diagonal approximants [M/M] vanish if the coefficients of η with the highest power in the numerator vanishes. Choosing the coefficients of the highest power of η equal to zero, we get a polynomial equations in α which can be solved very easily by using the built in utilities in the most manipulation languages such as Maple 9 and Mathematica 7. To apply Padé approximants it is useful to use the following transformation

$$\eta^{1/2} = x \quad (36)$$

into Eq. (35) to obtain the series free of fraction powers of x

$$f(\eta) = 1 + \alpha x^2 + \frac{4x^3}{3} + \frac{x^6}{3} + \frac{2\alpha x^5}{5} + \frac{2\alpha x^8}{15} + \frac{2x^9}{27} + \frac{3\alpha^2 x^7}{70} + ... \quad (37)$$

The diagonal Padé approximants can be applied in order to study the mathematical behavior of the potential $f(\eta)$ and to determine the initial slope of the potential $f'(0)$.

Table 1, clearly elucidates that Present solution method namely MLPDM shows excellent agreement with the solutions already available in literature [6-8]. This analysis shows that LDM suits for Thomas-Fermi equation.

Table 1: Comparison of the MLPDM with different analytical techniques for initial slope of the potential

Padé approximants	MLPDM	ADM [6]	VIM [7]	DTM [8]
[2/2]	-1.211413	-1.214140	-1.213802	-1.211413
[4/4]	-1.550525	-1.550526	-1.552671	-1.550525
[7/7]	-1.586019	-1.586021	-1.587245	-1.586021
[8/8]	-1.588076	-1.588076	-1.588076	-1.588076
[10/10]	-1.588076	-1.588076	-1.588076	-1.588076

CONCLUSION

The main aim of this work is to provide the series solution of the Thomas-Fermi equation by using modified Laplace Padé decomposition method (MLPDM). The new Modified Laplace Padé Decomposition Method (MLPDM) is a powerful tool to search for solutions of various nonlinear problems. The method overcomes the difficulty in other methods because it is efficient. We derived fast convergent results by combining the series obtained by modified Laplace decomposition method, with the diagonal Padé approximants. The convergence of MLPDM is also shown in Table 1. The analysis given here shows further confidence on MLPDM.

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