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ELMRES for Solving Ill-Conditioned Linear Equations

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Abstract: Most of the krylov subspace methods use some projection techniques to solve linear system of equations. ELMRES is a krylov subspace method which uses an oblique projection technique. This method transfers the original linear system into an upper least square problem by using the Hessenberg decomposition algorithm and update the current approximation iterate by the solution of this least square. Here the algorithm of ELMRES is described and it is applied to solve some popular ill-posed linear system of equations by using Tikhonov regularization technique.

Key words: ELMRES . Oblique projection . Hessenberg algorithm . least square problems

INTRODUCTION

The solution of large linear discrete ill-posed problem Ax = b where A and b are contaminated by noise, by iterative methods has recently received considerable attention. Due to the severe ill-conditioning of A and b, the meaningful solution of this equation is not yield, obviously. This paper considers the solution of linear systems of equations

$$Ax = b,\ \ A \in R^{n\times n},\ \ x, b \in R^n \tag{1}$$

with a large scale matrix of ill-determined rank. In particular, A is severely ill-conditiond or may be singular. These kinds of problems are obtained by discritizing of some ill-posed problems, such as Fredholm integral equations of the first kind with a smooth kernel, as well as in image debluring.

Because of the error in the right hand side vector b and the severe ill-conditioning of A, the least-square solution of minimal Euclidean norm of (1), given by $\hat{x}$, is not a meaningful approximation. Then the linear system (1) is replaced by nearby system that is less sensitive to perturbations.

Tikhonov regularization [4] in its simplest form replaces (1) by the minimization problem

$$\min_{x\in R^n}\left\{\|Ax-b\|^2+\lambda\|x\|^2\right\} \tag{2}$$

where $\lambda>0$ is a regularization parameter to check the norm of approximate solution. Determining the exact value of λ is not easy; however there exist some studies on this matter. It is proved that the solution of (2) is obtained by solving the linear system

$$(A^TA+\lambda I_n)x = A^Tb \tag{3}$$

Fortunately, equation (3) is a linear system which is not so sensitive to perturbations. On the other hand, the solution of (3) is also the answer of (1) approximately. Then it is acceptable to solve less sensitive and more well-conditioned linear systems (3) instead of finding the solution of Ax = b. Because of the above reasons, this Tikhonov regularization technique is applied to solve (nearly) singular linear equations.

ELMRES IMPLEMENTATION

Using iterative methods for solving equations Ax = b with a large scale matrix A has some important advantages like being economical in point of arithmetic costs and etc. then many researchers have focused to solve (1) by variety of these methods and have tried to modify them. Among these methods, projection techniques are much more approved. Because these methods modify the last approximation by solving some new problems with too lower dimensions.

ELMRES is an iterative method which uses an oblique projection technique. It was propused by Howell and Stephens in [5]. ELMRES algorithm is run under the Hessenberg column by column reduction. Then the Hessenberg reduction algorithm is recalled here.

Algorithm 1: Hessenberg decomposition algorithm

1. Given vector r and set $\beta = r(1)$ and $v_1 = r/\beta$.
2. for $j = 0,1,\ldots, k-1$ do
   a. $w = Av_{j+1}$, $u = w\,(2:j+1)$, $w = w - u^T\left[v_2 \ldots v_{j+1}\right]$,

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b. $h_{j+1}(1:j+2) = w(1:j+2)$, .
$w(1:j+2) = 0$, $v_{j+2} = w / h_{j+1}(j+2)$
End
3. Set $\bar{H}_k = [h_1\, h_2 \ldots h_k]$ then $\bar{H}_k \in R^{(k+1)\times k}$.

ELMRES implementation is similar to GMRES [8] because GMRES by the Arnoldi process reduces the matrix A while ELMRES do it by the above algorithm and both of them need to solve an upper hessenberg least square problem to modify their last iterations. In Algorithm 1, Constructing the vectors v_i, they are computed such that

$$v_i \perp \{e_1, e_2, \ldots, e_{i-1}\},\ i = 1,2,\ldots,k-1.$$

The cost to create k bases vectors of $K_k(A,r_0)$ is $2k^2n$ for GMRES while it is $k^2n\text{-}k^3/3$ for ELMRES. Then ELMRES is preferred in point of operators cost.

By setting $L_i = I_n + v_i e_i^T$ and $L = L_1L_2 \ldots L_k$ then $L_i^{-1} = I_n - v_i e_i^T$ and $L^{-1} = L_k^{-1}L_{k-1}^{-1} \cdots L_1^{-1}$ are obtained. By choosing $\hat{L}_k$ as the first k columns of L we have

$$A\hat{L}_k = \hat{L}_{k+1}\bar{H}_k \qquad (4)$$

Now, computing the least square $\min_{y \in C^k} \|\bar{H}_k y - \beta e_1\|$, the next iterate of ELMRES is obtained as:

$$x_k = x_0 + \hat{L}\bar{y}$$

where $\bar{y}$ is the solution of the least square problem and x_0 is an initial vector.

To know more about ELMRES, its algorithm is written in below

Algorithm 2: ELeMentary RESidual (ELMRES) Algorithm

1. Given x_0, set $r_0 = b\text{-}Ax_0$
2. Run the Hessenberg decomposition algorithm with r_0.
3. Solve the least square problem $\min_{y \in R^k} |\beta e_1 - \bar{H}_k y|$
4. Let $\bar{y} = \operatorname{argmin} \|\beta e_1 - \bar{H}_k y\|$, the next approximation iterate of the solution of (1) is $x_k = x_0 + L_k\bar{y}$
5. If x_k does not satisfy, set $x_0 = x_k$ and go to 1.

In ELMRES algorithm the approximation iterate x_k is computed such that

$$x_k = \operatorname*{argmin}_{x \in K_k(A,r_0)} \|L^{-1}(b - Ax)\|$$

The following theorem describes, How x_k minimizes $\|L^{-1}(b\text{-}Ax)\|$ on $K_k(A,r_0)$.

Theorem 1: Iterative solution x_k of ELMRES is the solution of minimizing problem

$$\min_{x \in K_k(A,r_0)} \|L^{-1}(b - Ax)\|$$

Proof: Let $x_k = x_0 + \hat{L}y,\ y \in R^k$ be the approximate solution of Ax = b. By multiplying the both sides of this relation by A, it is given that $Ax = Ax_0 + A\hat{L}y$. Then

$$b - Ax = b - Ax_0 - A\hat{L}y = r_0 - A\hat{L}y$$

Now, multiplying from the left by L^{-1}, the following relation is obtained

$$L^{-1}(b - Ax) = L^{-1}r_0 - L^{-1}A\hat{L}y = \begin{pmatrix} \beta e_1 \\ 0 \end{pmatrix} - \begin{pmatrix} \bar{H}_k \\ 0 \quad \alpha\hat{v}_{k+1} \end{pmatrix} y$$

where $\bar{H}_k \in R^{(k+1)\times k}$.
If

$$\bar{y} = \operatorname*{argmin}_{y \in C^k} \|\bar{H}_k y - \beta e_1\|$$

then next iterate $x_k = x_0 + \hat{L}\bar{y}$ obtained by ELMRES. Using (4), it is concluded that x_k minimizes $\|L^{-1}(b\text{-}Ax)\|$ on the subspace $K_k(A,r_0)$.

In Hessenberg decomposition algorithm, when $\beta = h_j(j+1)$ be zero then this reduction process will be stagnated. To avoid this stagnation, partial pivoting was proposed such that the pivoting element β chose as follows $\beta = \max_{k=j+1:n} h_j(k)$.

By this selection, the ELMRES algorithm does not breakdown [5].

NUMERICAL TESTS

In this section, ELMRES has tested for solving some popular ill-posed problems. For this important, the linear system of equation has selected from the "Regularization tools" package [4].

Fortunately, this package can be downloaded from the internet freely and it is well-known among the scientist such that they used these problem in their research. As it was discussed in the first section, regularization techniques should be used for solving ill-posed problems.

In [6] two possible Tikhonov implementations for solving ill-posed problems by GMRES were proposed.

Firstly, the solution of (2) or its equivalence (i.e. equation (3)) is computed instead of solving (1) directly. In second case, two steps should be done. The following least square problem

$$\min\left\{\left\|\bar{H}_k y-\beta e_1\right\|^2+\lambda\|y\|^2\right\}, \quad y\in R^k \tag{5}$$

be solved and after that, let $\bar{y}$ be the solution of (5), then the next approximation iterate of (1) will be computed as $x_k = x_0 + v_k\bar{y}$. To apply the first idea for ELMRES, a meaningful approximation x_k for (1) can be computed but unfortunately, the second idea is not applicable for ELMRES, practically.

Then in this experimental work, the first regularization technique has been chosen. To have a more reasonable test, ELMRES implementation has compared with GMRES, LSQR and QMR because these iterative methods are more popular among scientists. GMRES is a powerful method which converges to the solution quickly. LSQR [7] is a Krylov method with low convergence speed but the norms of residuals are usually decreased. QMR [3] is a fast iterative method in which it sometimes converges to the solution faster than GMRES. These methods were examined by different examples but in this paper, just two examples have been mentioned.

For simplicity, the regularization parameter $\lambda = 10^{-8}$, the restarted number k = 15 and the tolerance eps = 10^{-13} for these experiments have been considered.

Example 1: The Fredholm integral equation of the first kind,

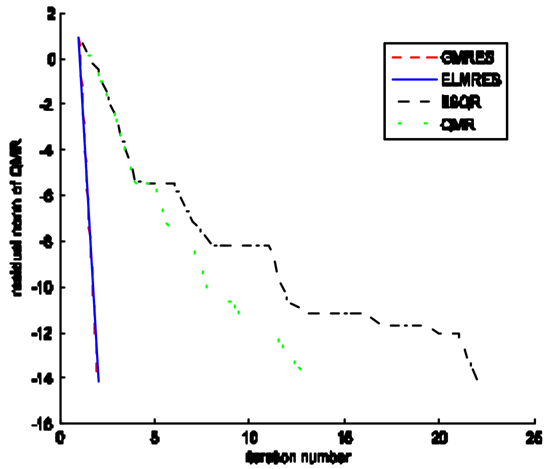
$$\int_0^{\pi/2} K(s,t)x(s)ds = b(t), \quad 0\le t\le \pi \tag{6}$$

With kernel K(s,t) = exp (s cos(t)), right hand side b(t) = 2 sin h(t)/t and solution x(t) = sin(t) is discussed by Baart [1]. We use the method code baart from [4] to discretize (6) by a Galerkin method with 200 orthonormal box function as test and trial function. The code produces the matrix $A\in R^{200\times 200}$ and a scaled discrete approximation of $x(\tau)$. Such that $x\in R^{200}$ is the solution of Ax = b.

The numerical comparisons for solving this problem by the mentioned algorithms are in the Table 1 and graph.

Table 1: Numerical results for Baart200

Baart200	Error	Iterate
ELMRES	6.5624e-015	2
GMRES	7.2717e-015	2
LSQR	7.1975e-015	22
QMR	1.1995e-014	13



Graph 3.2: Numerical residual norms of iterative methods for solving Baart200

According to Table 1, ELMRES and GMRES have been reached to the solution after two iterations with the residual norms 6.5624e-015 and 7.2717e-015 respectively. QMR was converged in 13 iterations while LSQR was closed to the exact solution after 22 iterations.

The norm of residuals of iterative methods in log10 is displayed in below.

From this graph, it is concluded that ELMRES and GMRES have been converged too faster than the rest methods while QMR and LSQR also converged quickly.

Example 2: The discretization of a new Fredholm integral equation of the first kind like (6) with both integration intervals equal to [0,1], with kernel K(s,t) = $(s^2+t^2)^{1/2}$, right hand side

$$b(t)=\frac{1}{3}\left((1+t^2)^{3/2}-t^3\right)$$

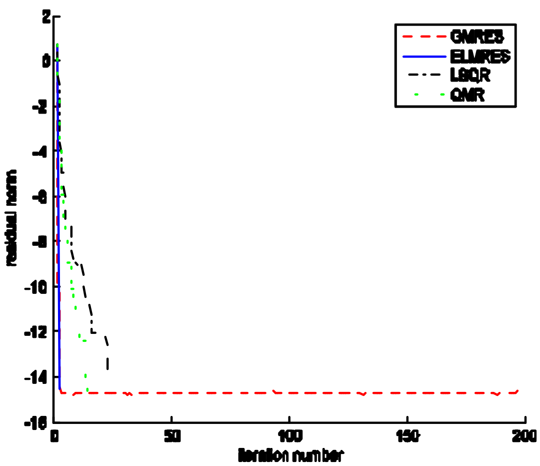
and with the solution x(t) = t (is discussed by Baker [2]) in Matlab code foxgoog from [4] with n = 200 has been selected which is a severely ill-posed problem.

The numerical results for solving this problem by the above methods are shown in Table 3.

In this test, ELMRES converged quickly, while GMRES at first had a similar behavior but it reached to

Table 3: Numerical results for Foxgood200

Foxgood200	Error	Iterate
ELMRES	2.7805e-015	2
GMRES	2.0275e-015	200
LSQR	1.7176e-014	23
QMR	2.2596e-015	14



Graph 3.4: Numerical residual norms of iterative methods for solving Baart200

the solution after 200 iterations such that a stagnation was happened from x_2 to x_{200}. Again, QMR obtained the solution in less repetitions rather than LSQR, but it converged slower than ELMRES. The norm of residuals of iterative methods for solving ill-posed linear system Foxgood200 is shown as follows.

From Figure 3.4, it can be seen that GMRES was converged to the solution quickly (like ELMRES) in first cycle, but later on ELMRES was faster.

CONCLUSION

ELMRES is an oblique projection method which can be used to solve ill-posed linear systems. In many cases, the convergence speed of ELMRES and GMRES are approximately equal while ELMRES needs less arithmetic computations. Numerical results of previous section confirm that ELMRES can be used to solve Ax = b problems. Anyhow, ELMRES and GMRES are more powerful than LSQR. Behavior of ELMRES and GMRES are approximately similar while ELMRES sometimes are reached to the approximate solution (1) in less iterates. Then this method is suggested to use for solving linear system of equations.

REFERENCES

1. Baart, M.L., 1982. The use of auto-correlation for pseudo-rank determination in noisy ill-conditioned least-squares problems. IMA J. Numer. Anal., 2: 241-247.
2. Baker, C.T.H., 1977. The Numerical Treatment of Integral Equations. Clarendon Press, Oxford, pp: 665.
3. Freund, R. and N. Nachtigal, 1991. QMR: A Quasi-Minimal Residual Method for Non-Hermitian Linear Systems. Numer. Math., 60: 315-339.
4. Hansen, P.C., 2007. Regularization Tools Version 4.0 for Matlab 7.3. Numerical Algorithms, 46: 189-194.
5. Howell, G. and D. Stephens, 2000. ELMRES: an oblique projection solver for sparse systems of linear equations. Technical Report, Florida Institute of Technology.
6. Lewis, B. and L. Reichel, 2009. Arnoldi-Tikhonov regularization methods. J. Comput. Appl. Math., 226: 92-102.
7. Paige, C.C. and M.A. Saunders, 1982. LSQR: An algorithm for sparse linear equations and sparse least squares. ACM Trans. Math. Software, 8: 43-71.
8. Saad, Y. and M.H. Schultz, 1986. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Statist. Comput., 7: 856-869.