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Discrete Symmetries Analysis of Burgers Equation with Time Dependent Flux at the Origin

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Abstract: In this work, we consider the Lie's method of infinitesimal transformation groups and discrete symmetries method for Burgers equation with time dependent flux at the origin. Following the Lie's method of infinitesimal transformation groups we determine the symmetry reductions and similarity solutions of the governing equation. By applying discrete symmetries analysis we have obtained three groups of discrete symmetries which lead to new symmetry reductions and similarity solutions of our problem. The analytical solutions which are obtained using symmetries and discrete symmetries are summarized in table form.

Key words: Lie group . discrete symmetries . Burger equation

INTRODUCTION

Burgers equation is a nonlinear parabolic partial differential equation which is encountered in the mathematical modeling of turbulent fluid and shock waves and considered as one of the fundamental model equation in the fluid dynamics to describe the shock waves and traffic flows. Burgers [1] introduced a mathematical model to capture some features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. The generalized Burgers equation has been studied for its solutions by many authors Rao *et al*. [6], Sachdev *et al*. [7] and Vaganan and Kumaran [9]. Clothier *et al*. [2] presented analytical constant-flux solution to Burgers equation. The initial/boundary value problem for the Burgers equation on the semiline is solved with a flux-type boundary condition at the origin, Biondini and Lillo [4]. In 1993, Mittal and Singhal [3] presented a numerical approximation of the one dimensional Burgers equation, they truncated one dimensional Fourier expansion with time dependent coefficients and formulated as the approximated problem which consisted of a system of nonlinear ordinary differential equations for the coefficients. Kutluay *et al*. [5] proposed finite-difference solution and analytical solution of the finite-difference approximations based on the standard explicit method to the one-dimensional Burgers equation which arises frequently in the mathematical modelling used to solve problems in fluid dynamics. Ozis *et al*. [8] applied a finite element approach to find numerical solutions for Burgers equation which used a model problem in turbulence and shock wave theory. A numerical study is made for solving one dimensional time dependent Burgers equation with small coefficient of viscosity Kadalbajoo *et al*. [10].

Symmetry group method plays an important role in the analysis of differential equations. The primary objective of the group classification methods advocated by Sophus Lie is to find one-or several-parameter local continuous transformations leaving the equations invariant and then exploit those to obtain the so-called invariant or similarity solutions Ovsiannikov [11] and Olver [13]. Recently, there have been several generalizations of the classical Lie group method for symmetry reductions. Similarity solutions of Burgers equation were obtained by Tajiri *et al*. [12]. Vaganan and Askan [14] found Direct similarity analysis of generalized burgers equations. Ibrahim *et al*. [15] investigated the similarity reductions for problems of radiative and magnetic field effects on free convection and mass-transfer flow past a semi-infinite flat plate. Azad and Mustafa [16] presented symmetry classification problem for wave equation on sphere. Also, Vaganan and Kumaran [17] presented similarity solutions for the damped Burgers equation with time-dependent viscosity. Lie symmetry analysis is performed for the general Burgers equation Liu *et al*. [18]. Nadjafikhah and Chamazkoti [19] solved the problem of the group classification of the general Burgers equation. Lie point symmetry groups and new exact solutions of a (2+1) dimensional generalized Broer-Kaup system are obtained Wang and Tian [20].

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Discrete symmetries are useful for increasing the efficiency of numerical methods for solving differential equations, by reducing either the computational domain or (for spectral methods) the space of trial functions. They also arise as hidden symmetries of some boundary value problems Crawford *et al*. [21]. Therefore it is important to be able to find discrete symmetries in a systematic manner. Some discrete symmetries (such as reflections) may be found by inspection or by using an ansatz, Olver, [13]. However, it is not generally possible to calculate all discrete point symmetries, which are determined by a system of nonlinear partial differential equations. In some instances, it is possible to use computer algebra to reduce this system to a differential Grobner basis, which may be solved more easily than the original system Reid *et al*. [22].

The aim of the present work to obtain the analytical solutions of the Burgers equation with time dependent flux at the origin using Lie group method. Also by using the discrete point symmetries, we introduce new group of analytical solutions of our problem.

MATHEMATICAL SIMULATION

We consider the boundary value problem for Burgers equation with time dependent flux at the origin as following

$$u_t = u_{xx} + 2uu_x, u = u(x,t) \tag{1}$$

$$u(\infty,t) = \lim_{x\to\infty} h(x,t) \tag{2a}$$

$$u^2(0,t) + u_x(0,t) = g(t), \qquad t \ge 0 \tag{2b}$$

where h (x,t) and g (t) will determined to get similarity solutions.

SYMMETRIES

The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformation Olver [13], Regers and Ames [23]. We consider the one-parameter Lie group of infinitesimal transformation in (x, t, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u) + O(\varepsilon^2) \\ t^* &= t + \varepsilon\tau(x,t,u) + O(\varepsilon^2) \\ u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2) \end{aligned} \tag{3}$$

where ε is the group parameter. In order to determining the point transformation (3), under which (1) is an invariant, the corresponding vector field of the transformation is written by

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{4}$$

and the second prolongation is

$$\begin{aligned} X^{(2)} = X &+ \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{tt}\frac{\partial}{\partial u_{tt}} \end{aligned} \tag{5}$$

Equation (1) can be written as

$$\Delta = u_{xx} + 2\,u\,u_x - u_t = 0 \tag{6}$$

The infinitesimals ξ, τ, η of a differential equation Δ = 0 of the second order are calculate from the condition

$$X^{(2)}(\Delta) = 0, \qquad \text{when} \qquad \Delta = 0 \tag{7}$$

From Eq. (6) and (7), we get

$$\eta^{xx} + 2\,\eta^x\,u - \eta^t + 2\,\eta\,u_x = 0 \tag{8}$$

where

$$\eta^{J\chi_i} = D_{\chi_i}\eta^J - u_{J\chi_i}D_{\chi_i}\xi_i - u_{J\chi_j}D_{\chi_i}\tau_j, \quad i \ne j \tag{9}$$

Where χ_i refer to the independent variables. Conditions on the infinitesimals ξ, τ and η are determined by equating coefficients of like derivatives of monomials in u_x and u_t and higher derivatives by zero. This leads to a system of partial differential equations from which we can determine ξ, τ and η. Analysis of this system of equations leads to an explicit form of ξ, τ and η in the form

$$\begin{aligned} \xi(x,t,u) &= \frac{1}{2}(c_1x - 4c_4)t + \frac{1}{2}c_2\,x + c_5, \\ \tau(x,t,u) &= \frac{1}{2}c_1t^2 + c_2t + c_3 \\ \eta(x,t,u) &= -\frac{1}{4}(x + 2ut)c_1 - \frac{1}{2}c_2\,u + c_4 \end{aligned} \tag{10}$$

The infinitesimal generators are

$$\begin{aligned} X_1 &= \frac{1}{2}x\,t\,\partial_x + \frac{1}{2}t^2\,\partial_t - \frac{1}{4}(x + 2ut)\partial_u \\ X_2 &= \frac{1}{2}x\,\partial_x + t\partial_t - \frac{1}{2}u\partial_u \end{aligned} \tag{11}$$

$$X_3 = \partial_t, \quad X_4 = -2\,t\partial_x + \partial_u, \quad X_5 = \partial_x$$

DISCRTETE SYMMETRIES

In this section, we will derive the discrete symmetries of PDE (1), which has a five-dimensional Lie algebra L of point symmetry generators given in (11). We will follow the method presented in Hydon [24]. By the commutator relation

$$[X_i, X_j] = c_{ij}^k X_k, \quad i < j, \quad i,j = 1,2,\ldots,5 \tag{12}$$

we can get the nonzero commutators in the following form

$$\begin{aligned}
&[X_1, X_2] = -X_1\\
&[X_1, X_3] = -X_2\\
&[X_1, X_5] = \frac{1}{4}X_4\\
&[X_2, X_3] = -X_3\\
&[X_2, X_4] = \frac{1}{2}X_4\\
&[X_2, X_5] = -\frac{1}{2}X_5\\
&[X_3, X_4] = -2\,X_5
\end{aligned}$$

Therefore the nonzero structure constants c_{ij}^k are

$$\begin{gathered}
c_{12}^1 = -1,\, c_{13}^2 = -1,\, c_{23}^3 = -1,\, c_{24}^4 = \frac{1}{2},\\
c_{34}^5 = -2\,,\, c_{25}^5 = -\frac{1}{2},\, c_{15}^4 = \frac{1}{4}
\end{gathered} \tag{13}$$

Following Hydon (2000), the matrices C(j), j = 1,2,…,5 are

$$C(1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{4} & 0 \end{pmatrix}$$

$$C(2) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$C(3) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 2\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C(4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & -2\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C(5) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & 0\\ 0 & 0 & 0 & 0 & -\frac{1}{2}\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Exponentiating the matrices εC(j), we get the adjoint matrices by using the following relation

$$A(j,\varepsilon) = \exp\{\varepsilon C(j)\}, \quad j = 1,2,\ldots,5 \tag{15}$$

Then we have

$$A(1,\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ \varepsilon & 1 & 0 & 0 & 0\\ \frac{1}{2}\varepsilon^2 & \varepsilon & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -\frac{1}{4}\varepsilon & 1 \end{pmatrix}$$

$$A(2,\varepsilon) = \begin{pmatrix} e^{-\varepsilon} & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & e^{\varepsilon} & 0 & 0\\ 0 & 0 & 0 & e^{-\frac{1}{2}\varepsilon} & 0\\ 0 & 0 & 0 & 0 & e^{\frac{1}{2}\varepsilon} \end{pmatrix}$$

$$A(3,\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & \frac{1}{2}\varepsilon^2 & 0 & 0\\ 0 & 1 & -\varepsilon & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 2\varepsilon\\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(4,\varepsilon)=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2}\varepsilon & 0 \\ 0 & 0 & 1 & 0 & -2\varepsilon \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(5,\varepsilon)=\begin{pmatrix} 1 & 0 & 0 & \frac{1}{4}\varepsilon & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2}\varepsilon \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (16)$$

After applying the nonlinear constraints $c^n_{lm} b^l_i b^m_j = c^k_{ij} b^n_k$, Hydon [24] and using the adjoint matrices to factor out the Lie symmetries, we find three formulae of the matrix (b^l_i) that

$$B1=\begin{pmatrix} 0 & 0 & \frac{1}{b^1_3} & 0 & b^5_1 \\ 0 & -1 & -\frac{b^2_3}{b^1_3} & \frac{1}{4}\frac{b^5_1}{b^1_3} & \frac{1}{2}\frac{4b^4_3 - b^5_1 b^2_3 b^1_3}{b^1_3} \\ b^1_3 & b^2_3 & \frac{1}{2}\frac{(b^2_3)^2}{b^1_3} & b^4_3 & -\frac{2 b^4_3 b^2_3}{b^1_3} \\ 0 & 0 & 0 & 0 & b^5_4 \\ 0 & 0 & 0 & -\frac{1}{8}b^1_3 b^5_4 & \frac{1}{4}b^2_3 b^5_4 \end{pmatrix}$$

$$B2=\begin{pmatrix} b^1_1 & 0 & 0 & b^4_1 & 0 \\ b^1_1 b^2_3 & 1 & 0 & -\frac{1}{4}b^1_1 b^5_3 + \frac{1}{2}b^4_1 b^2_3 & -\frac{2b^4_1}{b^1_1} \\ \frac{1}{2}b^1_1 (b^2_3)^2 & b^2_3 & \frac{1}{b^1_1} & -\frac{1}{4}b^5_3 b^1_1 b^2_3 & b^5_3 \\ 0 & 0 & 0 & b^4_4 & 0 \\ 0 & 0 & 0 & -\frac{1}{4}b^4_4 b^2_3 & \frac{b^4_4}{b^1_1} \end{pmatrix}$$

$$B3=\begin{pmatrix} \frac{1}{2}\frac{(b^2_1)^2}{b^3_1} & b^2_1 & b^3_1 & -\frac{1}{4}\frac{b^5_1 b^2_1}{b^3_1} & b^5_1 \\ 0 & 1 & \frac{2b^3_1}{b^2_1} & -\frac{1}{8}\frac{(b^2_1)^2 b^5_3}{b^3_1} & \frac{1}{2}\frac{2b^5_1 + (b^2_1)^2 b^5_3}{b^2_1} \\ 0 & 0 & \frac{2b^3_1}{(b^2_1)^2} & 0 & b^5_3 \\ 0 & 0 & 0 & b^4_4 & -\frac{4 b^3_1 b^4_4}{b^2_1} \\ 0 & 0 & 0 & 0 & \frac{2 b^3_1 b^4_4}{(b^2_1)^2} \end{pmatrix}$$

Now we try to simplify the matrix B1 by multiplying it by the matrix $A(j,\varepsilon)$ and choosing ε appropriately as follows

$$B1A(2,\varepsilon)=\begin{pmatrix} 0 & 0 & \frac{1}{b^1_3}e^{\varepsilon} & 0 & b^5_1 e^{\frac{1}{2}\varepsilon} \\ 0 & -1 & -\frac{b^2_3}{b^1_3}e^{\varepsilon} & \frac{1}{4}\frac{b^5_1}{b^1_3}e^{-\frac{1}{2}\varepsilon} & \frac{1}{2}\frac{(4b^4_3 - b^5_1 b^2_3 b^1_3)}{b^1_3}e^{\frac{1}{2}\varepsilon} \\ b^1_3 e^{-\varepsilon} & b^2_3 & \frac{1}{2}\frac{(b^2_3)^2}{b^1_3}e^{\varepsilon} & b^4_3 e^{-\frac{1}{2}\varepsilon} & -\frac{2 b^4_3 b^2_3}{b^1_3}e^{\frac{1}{2}\varepsilon} \\ 0 & 0 & 0 & 0 & b^5_4 e^{\frac{1}{2}\varepsilon} \\ 0 & 0 & 0 & -\frac{1}{8}b^1_3 b^5_4 e^{-\frac{1}{2}\varepsilon} & \frac{1}{4}b^2_3 b^5_4 e^{\frac{1}{2}\varepsilon} \end{pmatrix} \quad (17)$$

So (by choosing $\varepsilon = \ln\left|b^1_3\right|$) this equivalence transformation enables us to replace b^1_3 by ±1. Similarly, post-multiplying B1 by $A(3,\frac{b^2_3}{b^1_3})$, $A(4,\frac{b^5_1}{2b^1_3})$ and $A(5,\frac{4b^4_3}{b^1_3})$ are equivalent to setting $b^2_3 = 0$, $b^5_1 = 0$ and $b^4_3 = 0$. So, the matrix B_1 takes the following form

$$Bl=\begin{pmatrix} 0 & 0 & \delta & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_4^5 \\ 0 & 0 & 0 & -\frac{\delta}{8}b_4^5 & 0 \end{pmatrix}, \quad \delta=\pm 1 \qquad (18)$$

Similarly the other two matrices B_2, B_3 take the following formulae

$$B2=\begin{pmatrix} b_1^1 & 0 & 0 & b_1^4 & 0 \\ b_1^1 b_3^2 & 1 & 0 & \frac{1}{2}b_1^4 b_3^2 & -\frac{2\,b_1^4}{b_1^1} \\ \frac{1}{2}b_1^1(b_3^2)^2 & b_3^2 & \frac{1}{b_1^1} & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -\frac{1}{4}\beta b_3^2 & \frac{\beta}{b_1^1} \end{pmatrix}$$

$$B3=\begin{pmatrix} \frac{1}{2}\frac{(b_1^2)^2}{\alpha} & b_1^2 & \alpha & 0 & 0 \\ 0 & 1 & \frac{2\alpha}{b_1^2} & 0 & 0 \\ 0 & 0 & \frac{2\alpha}{(b_1^2)^2} & 0 & 0 \\ 0 & 0 & 0 & b_4^4 & -\frac{4\alpha b_4^4}{b_1^2} \\ 0 & 0 & 0 & 0 & \frac{2\alpha b_4^4}{(b_1^2)^2} \end{pmatrix}$$

The inequivalent discrete symmetries are those solutions of the following system

$$X_i\,\hat{x}_j = B\,\xi_j(\hat{x},\hat{t},\hat{u}), \quad i=1,2,\ldots,5, \quad j=1,2,3 \qquad (19)$$

where B = B_1, B_2, B_3, $\hat{x}_1=\hat{x}$, $\hat{x}_2=\hat{t}$, $\hat{x}_3=\hat{u}$ and $\xi_1=\xi$, $\xi_2=\tau$, $\xi_3=\eta$

First we consider $B=B_1$. By substituting from (10) and (18) into (19) we obtain the following system

$$\begin{pmatrix} X_1\hat{x} & X_1\hat{t} & X_1\hat{u} \\ X_2\hat{x} & X_2\hat{t} & X_2\hat{u} \\ X_3\hat{x} & X_3\hat{t} & X_3\hat{u} \\ X_4\hat{x} & X_4\hat{t} & X_4\hat{u} \\ X_5\hat{x} & X_5\hat{t} & X_5\hat{u} \end{pmatrix} = \begin{pmatrix} 0 & \delta & 0 \\ -\frac{1}{2}\hat{x} & -\hat{t} & \frac{1}{2}\hat{u} \\ \frac{1}{2}\delta\hat{x}\hat{t} & \frac{1}{2}\delta\hat{t}^2 & \delta(-\frac{1}{4}\hat{x}-\frac{1}{2}\hat{u}\hat{t}) \\ b_4^5 & 0 & 0 \\ \frac{1}{4}\delta b_4^5\hat{t} & 0 & -\frac{1}{8}\delta b_4^5 \end{pmatrix} \qquad (20)$$

The system (20) of first-order PDEs has the following general solution

$$\hat{x}=-\frac{1}{2t}b_4^5\,x,\ \hat{t}=-\frac{2\delta}{t},\ \hat{u}=-\frac{1}{8}\delta b_4^5\,x-\frac{1}{4}\delta b_4^5 u\,t \qquad (21)$$

where $b_4^5\neq 0$, we can determined this constant by the invariant condition

$$\hat{u}_{\hat{x}\hat{x}}+2\,\hat{u}\,\hat{u}_{\hat{x}}-\hat{u}_{\hat{t}}=0$$

when

$$u_{xx}+2\,u\,u_x-u_t=0 \qquad (22)$$

It turns out the following constraints

$$\delta=1, \qquad b_4^5=2\sqrt{2}\,\beta, \qquad \beta=\pm 1$$

Therefore, the first group of discrete symmetries is

$$\Gamma_1:(\hat{x},\hat{t},\hat{u})\mapsto(-\frac{\beta\sqrt{2}}{t}x,\ -\frac{2}{t},\ -\frac{\beta\sqrt{2}}{4}(x+2ut)) \qquad (23)$$

The general solution of the system (19), where B= B2 is as follows

$$\hat{x}=-\frac{2(4b_1^4+\beta x)}{b_1^1(b_3^2 t-2)}$$

$$\hat{t}=-\frac{2t}{b_1^1(t\,b_3^2-2)}$$

$$\hat{u}=-\frac{1}{2}\beta b_3^2 t\,u-b_1^4 b_3^2-\frac{1}{4}b_3^2\beta x+\beta u \qquad (24)$$

By substituting this result into the invariant condition (22), we obtain

$$b_3^2=0,\ b_1^1=1,\ \beta=\pm 1,\ b_4^1=\lambda$$

where λ is arbitrary constant. The second group of discrete symmetries

$$\Gamma_2:(\hat{x},\hat{t},\hat{u})\mapsto(4\lambda+\beta x, t, \beta u)) \qquad (25)$$

Also, the general solution of the system (19), where B = B3 is as follows

$$\hat{x}=\frac{2\,\alpha\,b_4^4\,x}{(b_1^2)^2},\ \hat{t}=-\frac{2\,\alpha(t-b_1^2)}{(b_1^2)^2},\ \hat{u}=b_4^4\,u \qquad (26)$$

Using the invariant condition (22) with (26) we obtain

$$b_4^4 = A,\ b_1^2 = \beta\sqrt{2}A,\ \alpha = 1$$

where A is an arbitrary constant. Therefore, third group of discrete symmetries is

$$\Gamma_3 : (\hat{x},\hat{t},\hat{u}) \mapsto (\frac{1}{A}x, \frac{(t-\sqrt{2}\beta A)}{A^2}, Au)) \qquad (27)$$

NEW GENERATORS FROM THE DISCRETE SYMMETRIES

The three groups of discrete symmetries which are given in (23), (25) and (27) generate new infinitesimal generators which give new similarity transformations. The new similarity transformations give new similarity and analytical solutions.

By using the discrete symmetries given in Eq.(23). The generators of Eq.(1) which given in (11) become;

$$\left.\begin{aligned}
X_6 &= (\frac{1}{4}u - \frac{2}{t^3}x)\partial_x + (\frac{2}{t^4}x^2 + \frac{4}{t^4} + \frac{1}{2}u^2)\partial_t + \frac{1}{2}ut\partial_u,\\
X_7 &= ((\frac{1}{t^2} - \frac{1}{16})x - \frac{1}{8}ut)\partial_x - (\frac{x^2+4}{t^3} + \frac{1}{8}xu + \frac{1}{4}u^2 t)\partial_t - (\frac{1}{4}ut^2 + \frac{1}{8}xt)\partial_u,\\
X_8 &= \frac{2}{t^2}\partial_t,\\
X_9 &= (-\frac{4\beta\sqrt{2}}{t^2} - \frac{\beta\sqrt{2}}{4})\partial_x + (-\frac{\beta\sqrt{2}}{4}u + \frac{4\beta\sqrt{2}}{t^3}x)\partial_t - \frac{\beta\sqrt{2}}{2}t\partial_u,\\
X_{10} &= -\frac{\beta\sqrt{2}}{t}\partial_x + \frac{\beta\sqrt{2}}{t^2}x\partial_t.
\end{aligned}\right\} \qquad (28)$$

Also, the other two groups of discrete symmetries (25) and (27) transform the generators in (11) to the following generators

Table 1: The similarity transformations and the exact solutions for the generator X_1 and for the three groups of discrete symmetries

X_1	$X_1 = \frac{1}{2}xt\partial_x + \frac{1}{2}t^2\partial_t - \frac{1}{4}(x+2ut)\partial_u$	
Similarity transformation	$\mu = \frac{x}{t}, \quad u = -\frac{1}{2}\mu + \frac{1}{t}f(\mu)$	
Similarity representation	$f''(\mu) + 2f(\mu)f'(\mu) = 0$	
Similarity solution	$f(\mu) = \frac{1}{c_1}\tanh(\frac{\mu+c_2}{c_1})$	
Exact solutions	Symm.	$u(x,t) = -\frac{1}{2t}[x - \frac{2}{c_1}\tanh(\frac{x+c_2 t}{c_1 t})]$
	Γ_1:	$\eta = \frac{\sqrt{2}\beta}{2}x,\ u = \frac{1}{\sqrt{2}\beta}f(\eta)$
		$u(x,t) = \frac{\sqrt{2}}{2\beta c_1}\tanh(\frac{\sqrt{2}\beta x + 2c_2}{2c_1})$
	Γ_2:	$\eta = \frac{4\lambda+\beta x}{t},\ u = -\frac{1}{2\beta}\eta + \frac{1}{\beta t}f(\eta)$
		$u(x,t) = -\frac{1}{2\beta t}[4\lambda + \beta x - \frac{2}{c_1}\tanh(\frac{4\lambda+\beta x + c_2 t}{c_1 t})]$
	Γ_3:	$\eta = \frac{Ax}{t-\sqrt{2}\beta A},\ u = -\frac{1}{2A}\eta + \frac{A}{t-\sqrt{2}\beta A}f(\eta)$
		$u(x,t) = \frac{A}{2\sqrt{2}\beta A - 2t}[\frac{x}{A} - \frac{2}{c_1}\tanh(\frac{Ax + c_2(t-\sqrt{2}\beta A)}{c_1(t-\sqrt{2}\beta A)})]$

$$\left.\begin{aligned}
&X_{11}=(2\beta\lambda t+\frac{1}{2}xt)\partial_x+\frac{1}{2}t^2\partial_t-(\beta\lambda+\frac{1}{4}x+\frac{1}{2}ut)\partial_u,\\
&X_{12}=(2\beta\lambda+\frac{1}{2}x)\partial_x+t\partial_t-\frac{1}{2}u\partial_u,\\
&X_{13}=-2\beta t\,\partial_x+\beta\partial_u,\quad X_{14}=\beta\partial_x,\\
&X_{15}=\frac{1}{2A^4}(xt-\sqrt{2}\beta Ax)x\partial_x+\frac{1}{2A^6}(t^2-2\sqrt{2}\beta At+2A^2)\partial_t+\frac{1}{4}(2\sqrt{2}\beta Au-2ut-x)\partial_u,\\
&X_{16}=\frac{1}{2A^2}x\partial_x-\frac{1}{A^4}(\sqrt{2}\beta A-t)t\partial_t-\frac{1}{2}A^2u\partial_u,\quad X_{17}=\frac{1}{A^2}\partial_t,\\
&X_{18}=\frac{1}{A^3}(2\sqrt{2}\beta A-2t)\partial_x+A\partial_u,\quad X_{19}=\frac{1}{A}\partial_x.
\end{aligned}\right\}\tag{29}$$

We note that from the discrete symmetries we get another 13 generators.

Table 2: (For the generator X_2 and discrete symmetries)

X_2	$X_2=\frac{1}{2}xt\partial_x+t\partial_t-\frac{1}{2}u\partial_u$	
Similarity transformation	$\mu=\frac{\sqrt{t}}{x},\ u=\frac{1}{x}f(\mu)$	
Similarity representation	$\mu^3f''(\mu)+(4\mu^2-\frac{1}{2})f'(\mu)-2\mu^2f(\mu)f'(\mu)-2\mu f^2(\mu)+2\mu f(\mu)=0$	
Similarity solution	$f(\mu)=(2c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{1}{4\mu^2})\,c_1-2c_2\mathrm{KummerU}(c_1,\frac{3}{2},\frac{1}{4\mu^2})+2\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{1}{4\mu^2})\,c_1-(2c_1-1)\mathrm{KummerM}(c_1,\frac{3}{2},\frac{1}{4\mu^2}))/(c_2\,\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{1}{4\mu^2})+\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{1}{4\mu^2}))$	
Exact solutions	Symm.	$u(x,t)=(2c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{x^2}{4t})\,c_1-2c_2\mathrm{KummerU}(c_1,\frac{3}{2},\frac{x^2}{4t})+2\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{x^2}{4t})\,c_1-(2c_1-1)\mathrm{KummerM}(c_1,\frac{3}{2},\frac{x^2}{4t}))/x(c_2\,\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{x^2}{4t})+\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{x^2}{4t}))$
	Γ_1:	$u(x,t)=-\frac{x}{2t}+(2c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},-\frac{x^2}{4t})\,c_1-2c_2\mathrm{KummerU}(c_1,\frac{3}{2},-\frac{x^2}{4t})+2\mathrm{KummerM}(1+c_1,\frac{3}{2},-\frac{x^2}{4t})\,c_1-(2c_1-1)\mathrm{KummerM}(c_1,\frac{3}{2},-\frac{x^2}{4t}))/(\beta x(c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},-\frac{x^2}{4t})+\mathrm{KummerM}(1+c_1,\frac{3}{2},-\frac{x^2}{4t})))$
	Γ_2:	$u(x,t)=(2c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{(4\lambda+\beta x)^2}{4t})\,c_1-2c_2\mathrm{KummerU}(c_1,\frac{3}{2},-\frac{x^2}{4t})+2\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{(4\lambda+\beta x)^2}{4t})\,c_1-(2c_1-1)\mathrm{KummerM}(c_1,\frac{3}{2},\frac{(4\lambda+\beta x)^2}{4t}))/((4\lambda+\beta x)(c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{(4\lambda+\beta x)^2}{4t})+\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{(4\lambda+\beta x)^2}{4t})))$
	Γ_3:	$u(x,t)=(2c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A})\,c_1-2c_2\mathrm{KummerU}(c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A})+2\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A})\,c_1-(2c_1-1)\mathrm{KummerM}(c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A}))/(x(c_2\mathrm{KummerU}(1+c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A})+\mathrm{KummerM}(1+c_1,\frac{3}{2},\frac{1}{4}\frac{x^2}{t-\sqrt{2}\beta A})))$

Table 3: (For the generator X_4 and discrete symmetries)

X_4	$X_4 = -2\, t\partial_x + \partial_u$	
Similarity transformation	$\mu = t, \quad u(x,t) = -\dfrac{x}{2\mu} + f(\mu)$	
Similarity representation	$\mu f'(\mu) + f(\mu) = 0$	
Similarity solution	$f(\mu) = \dfrac{1}{\mu} c_1$	
Exact solutions	Symm.	$u(x,t) = -\dfrac{1}{2t}(x - 2c_1)$
	Γ_1:	$\eta = -\dfrac{2}{t},\ u = -\dfrac{\sqrt{2}}{\beta t} f(\eta)$,
		$u(x,t) = \dfrac{c_1}{\sqrt{2}\beta}$
	Γ_2:	$\eta = t,\ u = -\dfrac{4\lambda + \beta x}{2\beta\eta} + \dfrac{1}{\beta} f(\eta)$,
		$u(x,t) = -\dfrac{1}{2\beta t}(4\lambda + \beta x - 2c_1)$
	Γ_3:	$\eta = \dfrac{t - \sqrt{2}\beta A}{A^2},\ u = -\dfrac{x}{2A^2\eta} + \dfrac{1}{A} f(\eta)$
		$u(x,t) = \dfrac{A}{2(\sqrt{2}\beta A - t)}(\dfrac{x}{A} - 2c_1)$

Table 4: (For $\chi = X_3 + cX_5$ and the discrete symmetries)

χ	$\chi = \partial_x + c\,\partial_t$	
Similarity transformation	$\mu = x - ct,\ u(x,t) = f(\mu)$	
Similarity representation	$f''(\mu) + 2f(\mu)f'(\mu) + cf'(\mu) = 0$	
Similarity solution	$f(\mu) = \dfrac{1}{2c_1}[2\tanh(\dfrac{\mu + c_2}{c_1}) - c\,c_1]$	
Exact solutions	Symm.	$u(x,t) = \dfrac{1}{2c_1}[2\tanh(\dfrac{x - t + c_2}{c_1}) - c\,c_1]$
	Γ_1:	$\eta = -\dfrac{1}{t}(\sqrt{2}\beta x - 2c),\ u = -\dfrac{1}{2\beta t}(\beta x + 2\sqrt{2} f(\eta))$,
		$u(x,t) = -\dfrac{1}{2t}[x + \dfrac{2\sqrt{2}}{\beta c_1}\tanh(\dfrac{-\sqrt{2}\beta x + c_2 t + 2}{c_1 t}) - c\,c_1]$
	Γ_2:	$\eta = 4\lambda + \beta x - ct,\ u = \dfrac{1}{\beta} f(\eta)$
		$u(x,t) = \dfrac{1}{2\beta c_1}[2\tanh(\dfrac{4\lambda + \beta x - t + c_2}{c_1 t}) - c\,c_1]$
	Γ_3:	$\eta = \dfrac{1}{A^2}(Ax + (\sqrt{2}\beta A - t)c),\ u = \dfrac{1}{A} f(\eta)$
		$u(x,t) = \dfrac{1}{2Ac_1}[2\tanh(\dfrac{Ax - t + \sqrt{2}\beta A}{A^2 c_1}) - c\,c_1]$

REDUCTION TO ODEs & EXACT SOLUTIONS

In this section, we present the similarity transformations and exact solutions for some generators given from symmetries and discrete symmetries, as in the following tables.

BOUNDARY CONDITIONS

Now we will investigate the case which suitable with the condition (2a, b).

(1) For the generator X_1, we get

$$\mu = \frac{x}{t}, \quad u = -\frac{1}{2}\mu + \frac{1}{t} f(\mu) \tag{30}$$

The system (1), (2) transform to the following

$$f''(\mu) + 2 f(\mu) f'(\mu) = 0 \tag{31}$$

with the boundary conditions

$$\begin{aligned} & f(\infty) = A_1 \\ & f^2(0) + 2f'(0) = B_1 \end{aligned} \tag{32}$$

where

$$h(x,t) = \frac{A_1 - x}{2t}, g(t) = \frac{B_1 - 2t}{4t^2}$$

and A_1, B_1 are arbitrary constants

(2) For the generator X_1 with Γ_1: we get

$$\eta = \frac{\sqrt{2}\beta}{2} x, \; u = \frac{-x}{2t} + \frac{1}{\sqrt{2}\beta t}(\eta + t f(\eta)) \tag{33}$$

The system (1), (2) transform to the following

$$f''(\eta) + 2 f(\eta) f'(\eta) = 0 \tag{34}$$

with

$$\begin{aligned} & f(\infty) = \sqrt{2}\beta A_2 \\ & \frac{1}{2} f^2(0) + \frac{1}{2} f'(0) = B_2 \end{aligned} \tag{35}$$

where

$$\lim_{x \to \infty} h(x,t) = A_2, \; g(t) = B_2$$

and A_2, B_2 are arbitrary constants

(3) For the generator X_1 with Γ_3: we get

$$\eta = \frac{Ax}{t - \sqrt{2}\beta A}, \; u = -\frac{1}{2A}\eta + \frac{A}{t - \sqrt{2}\beta A} f(\eta) \tag{36}$$

The system (1), (2) transform to the following

$$f''(\eta) + 2 f(\eta) f'(\eta) = 0 \tag{37}$$

with

$$\begin{aligned} & f(\infty) = A_3 \\ & f^2(0) + f'(0) = B_3 \end{aligned} \tag{38}$$

Where

$$h(x,t) = \frac{A}{t - \sqrt{2}\beta A}\left(A_3 - \frac{x}{2A}\right)$$

$$g(t) = \frac{1}{(t - \sqrt{2}\beta A)^2}(B_3 A^2 - 2t + \sqrt{2}\beta A)$$

and A_3, B_3 are arbitrary constants

DISCUSSION AND CONCLUDING REMARKS

In this paper, we have found discrete symmetries of the Burgers equation with time dependent flux at the origin given in equations (1) and (2). By applying Lie group method, we have found the infinitesimals (10) and its similarity generators (11) for Eqn. (1). Furthermore we have got three groups of discrete symmetries which are Γ_1, Γ_2 and Γ_3 (see Eqns. (23), (25), (27)). The similarity transformations and the exact solutions for some generators and the corresponding discrete symmetries are presented in tables.

Table 1; contains the similarity transformation, the similarity representation and exact solution of Eqn. (1) for the generator X_1. Also, it contains the new similarity transformations and new exact solutions corresponding to the discrete symmetries Γ_1, Γ_2 and Γ_3.

Also, Table 2-4 contain the similarity transformations, the similarity representations and exact solutions of Eqn. (1), the new similarity transformations and new exacnt solutions corresponding to the discrete symmetries Γ_1, Γ_2 and Γ_3 for the generators X_2, X_4 and the combination of X_3+cX_5 respectively.

The original problem (1) with the boundary (2) has transformed to the ordinary differential equation (31) with the boundary condition (32) using the transformations (30). The other two similarity representations (34) and (37) are corresponding to the discrete symmetries. We have determined the unknown functions h(x,t) and g(t) in condition we get similarity reduction. The ordinary differential equation (31) with the boundary conditions (32) are solved numerical and the behavior of the solutions is shown in Fig. 1.

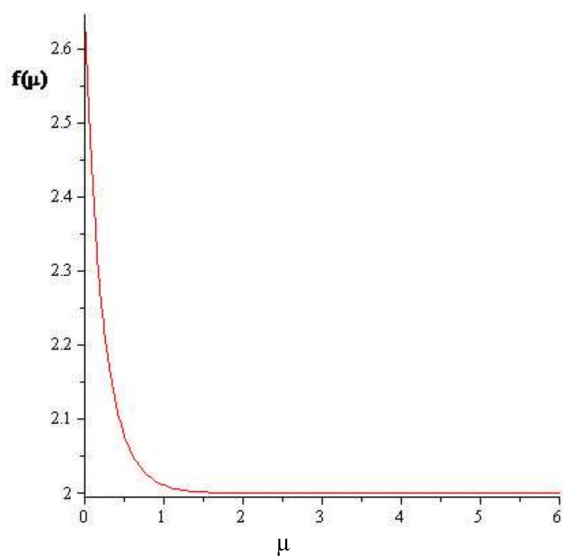

Fig. 1:

From Fig. 1 we see the function $f(u)$ decreases with increasing of the boundary layer thickness.

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