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Homotopy Perturbation Method for Fractional Differential Equations

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Abstract: In this paper, we apply Homotopy Perturbation Method (HPM) to solve nonlinear fractional differential equation. Numerical results explicitly reveal the complete reliability and efficiency of the proposed algorithm.

Key words: Homotopy perturbation method . fractional differential equation . truncated series method

INTRODUCTION

The fractional derivative has been occuring in many physical problems such as frequency dependent damping behavior of materials, motion of a large thin plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, the $PI^{\lambda}D^{\mu}$ controller for the control of dynamical systems etc. [1-4]. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order [5-11]. The solution of the differential equation containing fractional derivative is much involved. Most recently, applications have included classes of nonlinear Fractional Differential Equations (FDEs) [12] and their numerical solutions have been established by Diethelm and Ford [13]. Also, the solution of nonlinear fractional differential equation has been obtained through Adomian's decomposition method [12] and Variational iteration method [14]. He [15, 16] developed the Homotopy Perturbation Method (HPM) by merging the standard homotopy and perturbation. A wide class of physical problems [17-30] has been tackled by making an appropriate use of this algorithm. The basic inspiration of the prsent study is the extension of Homotopy Perturbation Method (HPM) for fractional differential equations [17]. It is observed that the proposed technique is highly suitable for such problems and numerical results clearly indicate the compelete reliability of the suggested HPM.

MATHEMATICAL DEFINITION

The mathematical definition of fractional calculus has been the main subject of many different approaches [17]. The left-sided Riemann-Liouville fractional integral of order $q>0$ of a function $f(x)$ is defined as:

$$\frac{d^{-q}f(x)}{dx^{-q}}=\frac{1}{\Gamma(q)}\int_0^x \frac{f(t)dt}{(x-t)^{1-q}},\ x>0$$

and the Riemann-Liouville's fractional derivative is defined as

$$\frac{d^{q}f(x)}{dx^{q}}=\frac{d^n}{dx^n}\left(\frac{d^{-(n-q)}f(x)}{dx^{-(n-q)}}\right)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dx^n}\int_0^x \frac{f(t)dt}{(x-t)^{1-n+q}}$$

where n is an integer that satisfies $n-1\le q<n$.

SOLUTION OF A NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

As an illustration of the present analysis, let us consider the HPM for solving the following nonlinear fractional differential equation:

$$\frac{du}{dt}+\frac{d^{1/2}u}{dt^{1/2}}-2u^2=0 \tag{1}$$

According to HPM, we construct the following homotopy

$$\frac{du}{dt}+p\{\frac{d^{1/2}u}{dt^{1/2}}-2u^2\}=0 \tag{2}$$

Assume the solution of Eq. (2) to be in the form:

$$u=u_0+pu_1+p^2u_2+p^3u_3+\ldots \tag{3}$$

Substituting (3) into (2) and equating the coefficients of like powers p, we get the following set of differential equations

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$$p^0 : \frac{du_0}{dt} = 0$$

$$p^1 : \frac{du_1}{dt} + \frac{d^{1/2}u_0}{dt^{1/2}} - 2u_0^2 = 0$$

$$p^2 : \frac{du_2}{dt} + \frac{d^{1/2}u_1}{dt^{1/2}} - 4u_0u_1 = 0$$

$$\vdots$$

Consequently, the first few terms of the HPM series solution are as follows:

$$u_0(t) = c$$

$$u_1(t) = \frac{2c\hbar\sqrt{t}}{\sqrt{\pi}} - 2c^2\hbar t$$

$$u_2(t) = \frac{2c\hbar\sqrt{t}}{\sqrt{\pi}} - 2c^2\hbar t + c\hbar^2 \left(\begin{array}{c} 2\frac{\sqrt{t}}{\sqrt{\pi}} - 2ct + t \\ -\frac{8ct^{3/2}}{\sqrt{\pi}} + 4c^2t^2 \end{array} \right)$$

$$\vdots$$

and so on. Hence, the HPM series solution is

$$u(t) = c + \left(2c^2t - 2c\frac{\sqrt{t}}{\sqrt{\pi}}\right) + \left(4c^3t^2 - \frac{8c^2t^{3/2}}{\sqrt{\pi}} + ct\right) + \dots \quad (4)$$

Now, consider the initial value problem for the nonlinear fractional differential equation

$$\frac{du}{dt} + \frac{d^{1/2}u}{dt^{1/2}} - 2u^2 = 0, u(0) = 1 \quad (5)$$

Therefore, the solution of the Eq. (4) is

$$u(t) = 1 + \left(2t - 2\frac{\sqrt{t}}{\sqrt{\pi}}\right) + \left(4t^2 - \frac{8t^{3/2}}{\sqrt{\pi}} + t\right) + \dots \quad (6)$$

VERIFICATION OF THE SOLUTION

We can look for the solution u(t) of the Eq.(5) in the form of the fractional power series:

$$u(t) = \sum_{i=0}^{\infty} u_i t^{1/2} \quad (7)$$

with $u_0 = c$, where c is a constant. By substituting (7) into (5) and comparing the coefficients of the results fractional power series, we obtain [12]:

$$u(t) = c - \frac{2c\sqrt{t}}{\sqrt{\pi}} + (c + 2c^2)t - \left(\frac{8c^2}{\sqrt{\pi}} + \frac{4c}{3\sqrt{\pi}}\right)t^{3/2} + \left[\frac{c}{2} + \left(5 + \frac{4}{\pi}\right)c^2 + 4c^3\right]t^2 - \left[\frac{8c}{15\sqrt{\pi}} + \left(\frac{32}{3\sqrt{\pi}} + \frac{64}{15\pi\sqrt{\pi}}\right)c^2 + \frac{352c^3}{15\sqrt{\pi}}\right]t^{5/2} + \dots \quad (8)$$

NUMERICAL EAMPLES AND DSCUSSIONS

Assuming c = 1 and retaining upto 5th power of t in Eq. (8), the truncated fractional power series becomes

$$u(t) = 1 - \frac{2\sqrt{t}}{\sqrt{\pi}} + 3t - \left(\frac{28}{3\sqrt{\pi}}\right)t^{3/2} + \left[\frac{19}{2} + \frac{4}{\pi}\right]t^2 - \left[\frac{104}{3\sqrt{\pi}} + \frac{64}{15\pi\sqrt{\pi}}\right]t^{5/2} + \dots \quad (9)$$

Table 1:Comparison between HPM-2nd order approximation and fractional power series solution (truncated upto 13-terms)

t (time)	HPM-2nd	Power series solution (13 terms)	Absolute error
0.00	1.000000	1.000000	0.000000
0.01	0.913049	0.912804	0.000245
0.02	0.889257	0.888937	0.000320
0.03	0.874706	0.874525	0.000181
0.04	0.864616	0.864758	0.000142
0.05	0.857224	0.857839	0.000615
0.06	0.851670	0.852877	0.001207
0.07	0.847467	0.849361	0.001894
0.08	0.844317	0.846971	0.002654
0.09	0.842021	0.845495	0.003474
0.10	0.840445	0.844794	0.004349

Table 2:Comparison between HPM-3rd order approximation and fractional power series solution (truncated upto 13-terms)

t t (time)	HPM-3rd	Power series solution (13 terms)	Absolute error
0.00	1.000000	1.000000	0.000000
0.01	0.912799	0.913142	0.000005
0.02	0.888954	0.889583	0.000017
0.03	0.874595	0.874525	0.000070
0.04	0.864911	0.864758	0.000153
0.05	0.858096	0.857839	0.000257
0.06	0.853251	0.852839	0.000374
0.07	0.849854	0.849361	0.000493
0.08	0.847574	0.846971	0.000603
0.09	0.846183	0.845495	0.000688
0.10	0.845522	0.844794	0.000728

Table 3:Comparison between HPM-4th order approximation and fractional power series solution (truncated upto 13-terms)

t t (time)	HPM-4th	Power series solution (13 terms)	Absolute error
0.00	1.000000	1.000000	0.000000
0.01	0.912803	0.913142	0.000001
0.02	0.888934	0.889583	0.000003
0.03	0.874517	0.874525	0.000008
0.04	0.864746	0.864758	0.000012
0.05	0.857827	0.857839	0.000012
0.06	0.852871	0.852839	0.000006
0.07	0.849369	0.849361	0.000008
0.08	0.846998	0.846971	0.000027
0.09	0.845539	0.845495	0.000044
0.10	0.844840	0.844794	0.000046

We compare the solution (4) by HPM with that of Eq. (8) by fractional power series method and the results are given below in Table 1-3.

From the above three tables we observe that approximate solution is in good agreement with truncated series solution. Of course the accuracy can be improved by computing more terms in the HPM.

CONCLUSION

The HPM is straightforward, without restrictive assumptions and the components of the series solution can be easily computed using any mathematical symbolic package. Moreover, this method does not change the problem into a convenient one for the use of linear theory. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems. When solutions are computed numerically, the rapid convergence is obvious. Moreover, no linearization or perturbation is required.

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