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Nonlinear Stability of the Cubic Functional Equation in Non-archimedean Random Normed Spaces

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Abstract: In this paper, the nonlinear stability of a functional equation in the setting of non-Archimedean normed spaces is proved. Furthermore, the interdisciplinary relation among the theory of random spaces, the theory of non-Archimedean space, the and the theory of functional equations are also presented

Key word: Hyers Ulam Rassias stability . cubic mappings . generalized normed space . Banach space . normed space

INTRODUCTION

The functional equation

$$f(2x+y)+f(2x-y)$$
$$=4f(x+y)+4f(x-y)+24f(x)-6f(y)$$

is said to be the quartic functional equation. Skof [1] by proving that if *f* is a mapping from a normed space X into a Banach space Y satisfying

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon$$

for some $\varepsilon>0$, then there is a unique quadratic function g: X→Y such that $\|f(x)-g(x)\| \leq \varepsilon/2$. Cholewa [2] extended Skofs theorem by replacing *X* by an abelian group *G*. Skof's result was later generalized by Czerwik [3] in the spirit of Hyers-Ulam-Rassias. The stability problem of the quadratic equation has been extensively investigated by a number of mathematicians and references therein. In addition, Alsina [4], Mihet¸ and Radu [5] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

In the sequel, the usual terminology, notations and conventions of the theory of random normed spaces shall be adopted, as in Schweizer and Sklar [6] and [7-10]. Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by

$$\Delta^+ = \left\{F : R \cup \{-\infty,+\infty\} \to [0,1]: F(0)=0 \quad \text{and} \quad F(+\infty) = 1\right\}$$

note that F is left continuous and non decreasing on R. Also the subset is the set

$$D^+ = \left\{ F\in \Delta^+ : l^- F(+\infty) = 1\right\}$$

Here $l^- f(x)$ denotes the left limit of the function *f* at the point

$$x, l^- f(x) = \lim_{t\to x} f(t)$$

The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., F≤G if and only if F(t)≤G(t) for all t in R. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0 \end{cases}$$

Definition 1: A mapping T: [0,1]×[0,1]→[0,1] is a continuous t-norm

If T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) T (a,1) = a for all a∈[0,1];
(d) T(a,b)≤T(c,d) whenever a≤c and b≤d and a,b,c,d∈[0,1].

Two typical examples of continuous t-norm are T(a,b) = ab and T(a,b) = min (a,b)
Now t-norms are recursively defined by $T^1 = T$ and

$$T^n(x_1,\ldots,x_{n+1}) = T\left(T^{n-1}(x_1,\ldots,x_n), x_{n+1}\right)$$

for n≤2 and $x_i \in [0,1]$, for all i∈{1,2,…,n+1}

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The t-norm T is Hadzic type if for given $\varepsilon\in(0,1)$ there is $\varepsilon\in(0,1)$ such that

$$T^{m}\left(1-\delta,\ldots,1-\delta\right)>1-\varepsilon,\quad m\in N$$

A typical example of such t-norms is $T(a,b)=\min(a,b)$.

Recall that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in [0,1], $T_{i=1}^{n}x_i$ is defined recursively by $T_{i=1}^{1}x_i=x_1$ and

$$T_{i=1}^{n}x_i=T\left(T_{i=1}^{n-1}x_i,x_n\right)$$

for $n\geq 2$. $T_{i=1}^{\infty}x_i$ is defined as $\lim_{n\to\infty}T_{i=1}^{n}x_i$

Definition 2: A non-Archimedean random normed space (briefly, non-Archimedean RN-space) is a triple (X, μ,T), where X is a nonempty set, T is a continuous t-norm and μ is a mapping from X into D^+ such that, the following conditions hold:

(PN1) $\mu_x(t)=\varepsilon_0(t)$ for all $t>0$ if and only if $x=0$;

(PN2) $\mu_{x-y}(t)=\mu_{y-x}(t)$ for all x,y in X and $t\geq 0$;

(PN3) $\mu_{\alpha x}(t)=\mu_x\left(\frac{t}{|\alpha|}\right)$ for all x in X, $\alpha\neq 0$ and $t\geq 0$;

(PN4) $\mu_{x+y}(\max\{t,s\})\geq T(\mu_x(t),\mu_y(s))$ for all $x,y,z\in X$ and $t,s\geq 0$.

Definition 3: Let (X,μ,T) be a non-Archimedean RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $t>0$ and $\varepsilon>0$, there exists positive integer N such that $\mu_{x_n-x}(t)>1-\varepsilon$ whenever $n\geq N$.

(2) A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every $t>0$ and $\varepsilon>0$, there exists positive integer N such that $\mu_{x_n,x_m}(t)>1-\varepsilon$ whenever $n\geq m\geq N$.

(3) A non-Archimedean RN-space (X,μ,T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 4: If (X,μ,T) is a non-Archimedean RN-space and $\{x_n\}$ is a sequence such that $x_n\to x$ then $\lim_{n\to\infty}\mu_{xn}(t)=\mu_x(t)$.

In this paper, the stability of the quadratic functional equation in the setting of non-Archimedean RN-space is established.

MAIN RESULTS

Definition 4: Let X, Y be vector spaces. The functional equation f: X→Y defined by

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \quad (1)$$

is called quartic functional equation.

Theorem 2: Let (X,ν,R) be non-Archimedean RN-space and (Y, μ, T) be a complete non-Archimedean RN-space. If f: X→Y be a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t)\geq\Psi(x,y,t)\quad(x,y\in X,t>0) \quad (2)$$

And

$$\Psi(2^{-k}x,2^{-k}y,t)\geq\Psi(x,y,\alpha t)\quad(x\in X,t>0) \quad (3)$$

$$\lim_{n\to\infty}\mathrm{T}_{j=n}^{\infty}M\left(x,\frac{\alpha^{j}t}{|2|^{kj}}\right)=1\quad(x\in X,t>0)$$

Then there exists a unique quartic mapping Q: X→Ysuch that

$$\mu_{f(x)-Q(x)}(t)\geq\mathrm{T}_{i=1}^{\infty}M\left(x,\frac{\alpha^{i+1}t}{|2|^{ki}}\right)$$

which

$$M(x,t):=T(\Psi(x,0,t),\Psi(2x,0,t),\ldots,\Psi(2^{k-1}x,0,t))\quad(x\in X,t>0)$$

Proof. Putting y = 0 in (2), then

$$\mu_{2f(2x)-32f(x)}(t)\geq\Psi(x,0,t)\qquad(x\in X,t>0)$$

Replacing x by 2x, then,

$$\mu_{f(2x)-16f(x)}(t) \geq \Psi(x,0,2t) \geq \Psi(x,0,t) \qquad (x \in X, t > 0)$$

Triangular inequality implies that

$$\mu_{f(2^{j+1}x)-16f(2^j x)}(t) \geq \Psi(2^j x, 0, t) \quad (x \in X, t > 0)$$

Thus

$$\begin{aligned}
\mu_{f(2^{j+1}x)-16^{j+1}f(x)}(t) &\geq T\left(\mu_{f(2^{j+1}x)-16f(2^j x)}(t), \mu_{16f(2^j x)-16^{j+1}f(x)}(t)\right) \\
&= T\left(\mu_{f(2^{j+1}x)-16f(2^j x)}(t), \mu_{f(2^j x)-16^j f(x)}\left(\frac{t}{|16|}\right)\right) \\
&\geq T\left(\mu_{f(2^{j+1}x)-16f(2^j x)}(t), \mu_{f(2^j x)-16^j f(x)}(t)\right) \\
&\geq T(\Psi(2^j x, 0, t), M_j(x,t)) \\
&= M_{j+1}(x,t)
\end{aligned}$$

Replacing x by 4x and triangular inequality implies that,

$$\begin{aligned}
\mu_{f(\frac{x}{2^{kn}})-64f(\frac{x}{2^{kn+k}})}(t) &\geq M\left(\frac{x}{2^{kn-k}}, t\right) \\
&\geq M(x, \alpha^{n-1}t) \quad (x \in X, t > 0, n = 0, 1, 2, \ldots)
\end{aligned}$$

Using the induction on n, is obtained that

$$\begin{aligned}
&\mu_{(2^{3k})^n f(\frac{x}{2^{kn}})-(2^{3k})^{n+1} f(\frac{x}{2^{kn+k}})}(t) \\
&\geq M\left(x, \frac{\alpha^{n+1}}{|(2^{3k})^n|}t\right) \\
&\geq M\left(x, \frac{\alpha^{n+1}}{|(2^3)^n|}t\right) \quad (x \in X, t > 0, n = 0, 1, 2, \ldots)
\end{aligned}$$

In order to prove convergence of the sequence $\left\{(2^{3k})^n f\left(\frac{x}{2^{kn}}\right)\right\}_{n \in N}$, replace x with $2^m x$ in (10) to find that for m,n>0

$$\lim_{n\to\infty} \mu_{(2^{3k})^n f(\frac{x}{2^{kn}})-Q(x)}(t) = 1 \quad (x \in X, t > 0)$$

To prove the uniqueness of the quadratic function, assume that there exists a quadratic function Q' which satisfies (4). Obviously $Q(2^n x) = 2^{3n}Q(x)$ and $Q'(2^n x) = 2^{3n}Q'(x)$ for all $x \in X$ and $n \in N$. Hence it follows from (2.4) that

$$\mu_{Q(x)-Q'(x)}(t) \geq T\left(\mu_{Q(x)-(2^{3k})^n f(\frac{x}{(2^k)^n})}(t), \mu_{(2^{3k})^n f(\frac{x}{(2^k)^n})-Q'(x)}(t), t\right)$$

for all $x \in X$. By letting $n \to \infty$, implies that the uniqueness of Q.

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