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Fixed Point Theory and Best Simultaneous Approximations

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Abstract: We use some theorem of fixed point theory in order to characterization of best simultaneous approximation elements of a subset of a normed space.

Key words: Best simultaneous approximation . KKM map . simultaneous proximinal

INTRODUCTION

The theory of best simultaneous approximation has been studied by many authors [1, 2, 5, 7, 9, 10]. Best simultaneous approximation is a generalization of best approximation in a sense. In this paper, we shall present some results about relation of best simultaneous approximations and KKM maps.

The structure of the paper is as follows. In Sec. 2 we give some definitions and preliminary results on best simultaneous approximation and fixed point theory. In Sec.3 we consider an inequality in relation with best simultaneous approximation and we provide some needs and assumptions for use a nonexpansive theorem. In Sec.4 we apply KKM-map to get some results about simultaneous approximation. We will use Ky Fan's theorem for this purpose. We will release convexity assumption in this section.

PRELIMINARIES

Let X be a normed linear space. For a non-empty W of X and a non-empty bounded set S in X define

$$d(S,W) = \inf_{w\in W} \sup_{s\in S} \|S - W\|$$

We recall [3, 7] that an element $w_0 \in W$ is called a best simultaneous approximation to S from W if

$$d(S,W) = \sup_{s\in S} \|s - w_0\|$$

The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$.

Definition 2.1: If for each bounded set S in X there exist at least one best simultaneous approximation from W to S from W, then W is called a simultaneous proximinal subset of X. If for each bounded set S in X there exists a unique best simultaneous approximation to S from W, then W is called simultaneous Chebyshev subset of X. [6] for the following definitions:

Definitions 2.2: A set C in a linear normed space X is said to be star-shaped if there is at least one $p\in C$ such that $(1-\lambda)p + \lambda x \in C$ for all $x\in C$ and $0<\lambda<1$. The point $p\in C$ is said to be the star center of C. Every convex set C is star-shaped but not conversely.

Definitions 2.3: Let $C\subseteq X$ be a subset of a linear normed space X. Then the convex hull of C is defined as co(C)

$$= \{\sum \lambda_i x_i : 0 \le \lambda_i \le 1, \sum \lambda_i = 1, x_i \in C\}.$$

Let (X,d) be a metric space and let CB(X) denote the family of all nonempty, closed bounded subsets of X. For A, $B\in CB(X)$, the Hausdorff metric, denoted by H(A,B), is defined by

$$H(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$$

When (X, d) is a complete metric space, then so is (CB(X), H). Let X and Y be two sets. A multifunction (set-valued map) F from X to Y, denoted by $F: X\to Y$, is a subset $F\subseteq X\times Y$. The value of F are the sets $F(x) = \{y\in Y: (x,y)\in F\}$. For $A\subseteq \overline{X}$ the set

$$F(A) = \cup_{x\in A} F(x) = \{y\in Y : F^{-1}(y)\cap A \ne 0\}$$

is called the image A under F.

A multifunction $F: X\to CB(X)$ is called nonexpansive if $H(F(x), F(y)) \le d(x,y)$.

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Let X be A linear normed space, W a nonempty subset of X and S a bounded set in X. In addition, let r = d(S,W). We consider the following inequality in this section:

$$\sup_{s\in S}\|S - W\| \le r + \|x - w\|$$

It is clear that if x = w be a solution for this inequality, then $x\in S_W(S)$. Define the multifunction $F:W\to 2^W$ by

$$F(x) = \{w \in W : \sup_{s\in S}\|s - w\| \le r + \|x - w\|\}$$

Lemma 2.1: Let X,W, S are as above and $x\in W$ such that $F(x) = \phi$. Then W is simultaneous Chebyshev and simultaneous proximinal set. (In other word if there exist $x\in W$ such that

$$r + \|x - w\| < \sup_{s\in S}\|s - w\|;\ \text{for all } w \in W$$

is hold. Then W is simultaneous Chebyshev and simultaneous proximinal set.)

Proof 2.2: Let $w_1\in W$ is arbitrary. Then by the definition of r and also by the suppose their exist $v_1\in W$ such that

$$\sup_{s\in S}\|s - v_0\| \le r + \|x - w_1\| < \sup_{s\in S}\|x - w_1\|$$

So we can define a sequence as the following.

Let w_1, arbitrarily, is selected and so we have v_1 as above. Now for $v_1\in W$ we have $v_2\in W$ such that

$$\sup_{s\in S}\|s - v_2\| \le r + \|x - v_1\| < \sup_{s\in S}\|x - v_1\|$$

With continue to this procedure we have a decreasing bounded sequence, $\{x-v_n\}$, in R. Therefore it is convergent to zero. we have

$$\sup_{s\in S}\|s - v_{n+1}\| < r + \|x - v_n\|$$

So when n tend to the infinite follows that $\sup_{s\in S}\|s - x\| \le r$. That is $x\in S_W(S)$. If

$$x,y \in S_W(S) \text{ and } x \ne y \text{ then}$$

$$r + \|x - y\| < \sup_{s\in S}\|s - y\| = r \Rightarrow \|x - y\| < 0$$

This is a contradiction and so W is a simultaneous Chebyshev set.

Lemma 2.3: Let X,W,S are as the above lemma. If $x\in S_W(S)$ then

$$r + \|x - w\| \le 3\sup_{s\in S}\|s - w\| \text{ for all } w \in W$$

Proof 2.4: For each $w\in W$ we have

$$r \le \sup_{s\in S}\|s - w\| \text{ and } \sup_{s\in S}\|s - x\| \le \sup_{s\in S}\|s - w\|$$

Hence

$$r + \|x - w\| \le r + \|x - s\| + \|w - s\| \le$$

$$r + \sup_{s\in S}\|s - x\| + \sup_{s\in S}\|s - w\| \le 3\sup_{s\in S}\|s - w\|$$

Let A is a nonempty subset of a metric space (X, d). We recall that the diameter of A, denoted diamA, is defined by diam A = sup $\{d(x,y): x,y\in A\}$

Corollary 2.5: Let X,W,S are as the above lemma. Then diam $S_W(S)\le 2r$.

Proof 2.6: If $x,y\in S_W(S)$ then, by the above lemma, we have $\|x-y\|\le 2r$. Now the state is clear.

In view of the lemma 2.1 we consider the following states:

Remark 2.1

1. $\exists x \in W; \forall w\in W; r + \|x - w\| \ge \sup_{s\in S}\|s - w\|$ (equivalently $F(x) = W$)
2. $\exists x \in W; \forall w\in W; r + \|x - w\| \le \sup_{s\in S}\|s - w\|$ (equivalently $F(x) = \phi$)
3. $\forall x \in W; \exists w \in W; \sup_{s\in S}\|s - w\| \le r + \|x - w\|$ (equivalently $F(x) \ne \phi$)
4. $\forall x \in W; \exists w \in W; r + \|x - w\| \le \sup_{s\in S}\|s - w\| \le r + \|x - w\|$ (equivalently $F(x) \ne W$)

In (1), (4) and in the some of the remained cases, similarly to the argue in the proof of the lemma 2.1, we can show that $x \in S_W(S)$ and so W is simultaneous proximinal set. (2) is the lemma 2.1. We consider 3 in the next section.

APPLICATION OF FIXED POINT THEORY

We need the following result.

Theorem 3.1: [6] Let X be a Banach Space, W a nonempty closed and star-shaped subset of X and F: $W \to 2^X$ a nonexpansive mapping such that $(x,y] \cap W \neq \phi$ for all $x \in W$ and $y \in F(x)$, where $(x,y] = \{(1-\alpha)x+\alpha y, 0<a\leq 1\}$. Further, assume that F(W) is bounded and (I-F)(W) is closed. Then F has a fixed point.

Lemma 3.2: Let W is a closed bounded set of linear normed space X such that the inequality 3, in the above remark, is hold. Let S is a bounded set of X and define

$$F(x) = \{w \in W : \sup_{s \in S} \|s - w\| \leq r + \|x - w\|\}$$

Then for each $x \in W$; $F(x) \in CB(X)$.

Proof 3.3: By the definitions of CB(X) and for each $x \in W$ we show that

- F(x) is nonempty.
- F(x) is bounded
- F(x) is closed

By the inequality 3, in the above remark, state (1) is true. Since $F(x) \subseteq W$ so (2) is true.

For (3): Let $\{w_n\} \subseteq F(x)$ and $w_n \to w_0$. We show that $w_0 \in F(x)$. Since W is closed so $w_0 \in W$. For each $s \in S$ we have

$$\|s - w_0\| \leq \|s - w_n\| + \|w_n - w_0\|$$

Therefore

$$\sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - w_n\| + \|w_n - w_0\|$$

So when n is sufficiently large we have

$$\sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - w_n\| \leq r + \|x - w_0\| + \|w_0 - w_n\|$$

So when $n \to \infty$ result that $\sup_{s \in S} \|s - w_0\| \leq r + \|x - w_0\|$
So hence $w_0 \in F(x)$ and F(x) is closed.

Proposition 3.4: Let S be a bounded subset of Banach space X,W a nonempty bounded, closed and star-shaped subset of X. Let

$$F(x) = \{w \in W : \sup_{s \in W} \|s - w\| \leq r + \|x - w\|\}$$

such that:

1. Inequality 3, in the above remark, is hold.
2. $\sup_{x \in W} H(F(x), S) \leq \alpha < \infty$
3. There exist a>0 such that

$$\text{if } \|x - y\| \leq a \text{ then } F(x) = F(y); x, y \in W$$

4. $2\alpha \leq a$
5. (I-F)(W) is closed.

Then W is a simultaneous proximinal set.

Proof 3.5: By the assumption (1) $F(x) \neq \phi$ and by the lemma 3. $F(x) \in CB(X)$ view to (2). Since $F(x) \subseteq W$ so clearly the condition 3 is hold. Let $x, y \in W$. If $\|x-y\| \leq a$ then F(x) = F(y) and therefore H (F(x), F(y)) = 0. So H (F(x), F(y)) = $0 \leq \|x-y\|$. If $a \leq \|x-y\|$ then by (4),

$$H(F(x), F(y) \leq H(F(x), S) + H(S, F(y)) \leq 2a \leq a \leq \|x - y\|$$

Therefore, by the theorem 3.1, F is a nonexpansive multifunction map and has a fixed point as $x \in F(x)$. That is $x \in S_W(x)$ and W is a simultaneous proximinal set.

Remark 3.1: Every one of the assumptions listed in the proposition may be replace with other various assumptions. For example if W be contained in a finite dimensional subspace or it is contained in a compact subset or, even, it is contained in a weakly compact subset of X then (I-F)(W) is closed. At least in L^P space condition (3) means that the set $\{w \in W: F(x) \neq F(y)\}$ is negligible set.

Example 3.1: If in the above proposition we assume that W is contained in a weakly compact subset of X, then (I-F)(W) is closed.

Proof 3.6: See chapter 1, theorem 1.92 of [6].

APLICATION OF KKM-MAP PRINCIPLE

Knaster, Kuratowski and Mazurkiewicz proved a very important result (KKM theorem) in 1929, presently it is known as the KKM-map principle. See example of KKM-maps in [4]. We give an example of KKM-map and apply it to the best simultaneous approximation.

Definition 4.1: [6] Let W be a nonempty subset of X. A map F: $W \to 2^X$ is called a KKM-map if $co\{x_1, \ldots, x_n\} \subseteq \cup_{i=1}^{n} F(x_i)$ for each finite subset $x_1, \ldots, x_n$ of X.

Observe that if F is a KKM-map, then $x \in F(x)$ for each $x \in X$. It is clear that F(x) in the previous section is not a KKM-map. The following well known result was established by ky Fan.

Theorem 4.1: [6] Let W be a subset of a Hausdorff topological vector space X and let T: $W \to 2^X$ be a closed-valued KKM-map. If $T(x_0)$ is compact for at least one $x_0 \in W$ then $\cap_{w \in W} T(w) \neq \phi$.

Lemma 4.2: Let X is a linear normed space, W is a closed and convex subset of X and S is a bounded subset of S such that $W \cap S = \phi$. Define G:$W \to 2^W$ by

$$G(x) = \{w \in W : \sup_{s \in S} \|s - w\| \leq \sup_{s \in S} \|s - x\|\}$$

Then G is a closed- valued KKM-map.

Proof 4.3: Let $x_1, \ldots, x_n \in W$ and $a \in co\{x_1, \ldots, x_n\}$. Then there are $a_1, \ldots, a_n \in [0,1]$ with $\sum_{i=1}^{n} a_i = 1$ such that $a = \sum_{i=1}^{n} a_i x_i$. For a suitable permutation of $x_1, \ldots, x_n$ which we show it again with $x_1, \ldots, x_n$ we have

$$\sup_{s \in S} \|s - x_1\| \leq \ldots \leq \sup_{s \in S} \|s - x_n\|$$

So

$$\|s - a\| = \left\|\sum_{i=1}^{n} \alpha_i s - \sum_{i=1}^{n} \alpha_i x_i\right\| \leq \sum_{i=1}^{n} \alpha_i \|s - x_i\| \leq$$

$$\sum_{i=1}^{n} \alpha_i \sup_{s \in S} \|s - x_n\| \leq \sup_{s \in S} \|s - x_n\| \sum_{i=1}^{n} \alpha_i = \sup_{s \in S} \|s - x_n\|$$

Therefore $\sup_{s \in S} \|s - a\| \leq \sup_{s \in S} \|s - x_n\|$. That is $a \in G(x_n)$ and therefore $a \in \cup_{i=1}^{n} G(x_i)$. Let $x \in W$. We show that G(x) is a closed subset of X. Let the sequence $\{w_n\}$ tend to w_0 and $w_n \in G(x_n)$. Since W is closed so $w_0 \in W$. For each $s \in S$ we have

$$\|s - w_0\| = \|s - w_0 + w_n - w_n\| \leq \|s - w_n\| + \|w_n - w_0\|.$$

Hence

$$\sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - w_n\| + \|w_n - w_0\|$$

$$\leq \sup_{s \in S} \|s - x\| + \|w_n - w_0\|$$

So when $w_n \to w_0$ follows that $\sup_{s \in S} \|s - w_0\| \leq \sup_{s \in S} \|s - x\|$. That is $w_0 \in G(x_n)$ and $G(x_n)$ is closed.

Now, by using of the theorem 4.1, we get some result from the lemma 4.2 as the following

Corollary 4.4: Let W is a closed and convex subset of linear normed space X and S is a bounded subset of X such that $W \cap S = \phi$. Define F:$W \to 2^W$ by F = (x) = ($w \in W$: $\sup_{s \in S} \|s - w\| \leq \sup_{s \in S} \|s - x\|$). If there exist $x_0 \in W$ such that $F(x_0)$ is compact then W is simultaneous proximinal set of X.

Proof 4.5: By the lemma 4.2 all condition of the theorem 4.1 are holds. Therefore $\cap_{x \in X} F(x) \neq \phi$. Let $w \in \cap_{x \in X} F(x)$ and $x \in W$ then, we have $x \in F(x)$. Hence for each $x \in W$, we have $\sup_{s \in S} \|s - w\| \leq \sup_{s \in S} \|s - x\|$. So

$$\sup_{s \in S} \|s - w\| = \inf_{w_0 \in W} \sup_{s \in S} \|s - w_0\| = r$$

That is $w \in S_W(S)W$ and W is a simultaneous proximinal set.

Corollary 4.6: Let W is a closed and convex subset of linear normed space X. If F(x), KKM-map in the lemma 4.2, is finitely closed. Then W is simultaneous proximinal set.

Proof 4.7: In fact it is an extension of theorm 4.1 [4]. By this extension, again, we have $\cap_{x \in X} F(x) \neq \phi$. Therefore, similarly argue in the above lemma, $S_W(S) \neq \phi$. (A subset $W \subseteq X$ is finitely closed if its intersection with each finite dimensional linear subspace $L \subseteq X$ is closed in the Euclidean topology of L.)

Remark 4.1: There are various extension of the theorem 4.1, presented, by Granas [4] and that all of them give information about simultaneous proximinal sets.

we have a trivial case that result also form 4.4:

$$F(x) = \{x\} \text{ if and only if } x \in S_W(S).$$

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