World Applied Sciences Journal 12 (12): 2175-2179, 2011
ISSN 1818-4952

Bipolar Fuzzy Subquasigroups

N. O. Alshehri

Department of Mathematics, Faculty of Sciences(Girls)
King Abdulaziz University, P.O. Box 126238, Jeddah, Saudi Arabia
nalshehrie@kau.edu.sa

Abstract: In this paper, we introduce the concept of bipolar fuzzy subquasigroups and investigate some of their important properties. Using bipolar fuzzy subquasigroups, characterizations of Artinian and Noetherian subquasigroups are also discussed.

Key words: Subquasigroups, Bipolar sets, Bipolar fuzzy subquasigroups, Artinian and Noetherian subquasigroups

INTRODUCTION

Zadeh [1] in 1965 first introduced the notion of a fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines such as engineering, medical science, social science, physics, statistics, graph theory, artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, expert systems, decision making, automata theory and so on. On the other hand, the concept of bipolar fuzzy sets was initiated by Zhang in [2, 3] as a generalization of fuzzy sets. Bipolar fuzzy sets are extensions of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to the interval $[-1, 1]$. In a bipolar fuzzy set, the membership degree 0 of an element means that the element is irrelevant to the corresponding property, the membership degree $(0, 1]$ of an element indicates that the element somewhat satisfies the property, and the membership degree $[-1, 0)$ of an element indicates that the element somewhat satisfies the implicit counter-property. Although bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other, they are essentially different to each other(see [4]).

In many domains, it is important to be able to deal with bipolar information. It is noted that positive information represents what is granted to be possible, while negative information represents what is considered to be impossible. This domain has recently motivated new research in several directions. In particular, fuzzy and possibilistic formalisms for bipolar information have been recently proposed [5]. Because when we deal with spatial information in image processing or in spatial reasoning applications, this bipolarity also occurs. For instance, when we assess the position of an object in a space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible places. As another example, let us consider the spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving some room for indetermination. This corresponds to the idea that the union of positive and negative information does not cover all the space. Akram and Dudek applied this concept to subquasigroup in [6] and studied some of its properties. The notion of new generalized fuzzy subquasigroups, $(\in, \in \vee q_m)$-fuzzy subquasigroups, was introduced in [7]. In this paper, we introduce the concept of bipolar fuzzy subquasigroups and investigate several important properties. We present some characterizations of bipolar fuzzy subquasigroups. Using bipolar fuzzy subquasigroups, characterizations of Artinian and Noetherian subquasigroups are also discussed. We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [8-20].

PRELIMINARIES

In this section we cite some elementary aspects that are necessary for this paper.
A groupoid $(G, \cdot)$ is called a *quasigroup* if for any a , b $\in G$ each of the equations $a \cdot x = b$, $x \cdot a = b$ has a unique solution in G. A quasigroup may be also defined as an algebra $(G, \cdot, \backslash, /)$ with three binary operations $\cdot$, $\backslash$, $/$ satisfying the following identities:

$$(x \cdot y)/y = x, \quad x \backslash (x \cdot y) = y,$$

$$(x/y) \cdot y = x, \quad x \cdot (x \backslash y) = y.$$

Such defined quasigroup is called an *equasigroup*.

A nonempty subset S of a quasigroup $\mathcal{G} = (G, \cdot, \backslash, /)$ is called a *subquasigroup* if it is closed with respect to these three operations. In this paper $\mathcal{G}$ always denotes an equasigroup $(G, \cdot, \backslash, /)$; G always denotes a nonempty set.

Definition 1. *Let G be a nonempty set. A bipolar fuzzy set B in G is an object having the form*

$$B = \{(x, \mu^+(x), \nu^-(x)) \mid x \in G\}$$

where $\mu^+ : G \to [0, 1]$ and $\nu^- : G \to [-1, 0]$ are mappings.

The positive membership degree $\mu^+(x)$ denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set B, and the negative membership degree $\nu^-(x)$ denote the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar fuzzy set B. If $\mu^+(x) \neq 0$ and $\nu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for B. If $\mu^+(x) = 0$ and $\nu^-(x) \neq 0$, it is the situation that x does not satisfy the property of B but somewhat satisfies the counter property of B . It is possible for an element x to be such that $\mu^+(x) \neq 0$ and $\nu^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of G.
For the sake of simplicity, we shall use the symbol $B = (\mu^+, \nu^-)$ for the bipolar fuzzy set $B = \{(x, \mu^+(x), \nu^-(x)) \mid x \in G\}$.

BIPOLAR FUZZY SUBQUASIGROUPS

Definition 2. *A bipolar fuzzy set $B = (\mu^+, \nu^-)$ is said to be a bipolar fuzzy subquasigroup of $\mathcal{G}$ if it satisfies the following conditions:*

(1) $\mu^P(x * y) \geq \min\{\mu^P(x), \mu^P(y)\}$,

(2) $\nu^N(x * y) \leq \max\{\nu^N(x), \nu^N(y)\}$

for all $x, y \in G$.

Example 3. *Let $G = \{0, a, b, c\}$ be a quasigroup with the following multiplication table:*

$\cdot$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $B = (\mu^P, \nu^N)$ be a bipolar fuzzy set in $\mathcal{G}$ defined by

	0	a	b	c
μ_A^P	0.8	0.06	0.06	0.06
μ_A^N	-0.7	-0.14	-0.14	-0.14

By routine computations, it is easy to see that B is a bipolar fuzzy subquasigroup of $\mathcal{G}$.

Definition 4. *Let $B = (\mu^P, \nu^N)$ be a bipolar fuzzy set and $(s, t) \in [-1, 0] \times [0, 1]$. We define:*

(a) *the sets $B_t^+ = \{x \in G \mid \mu^P(x) \geq t\}$ and $B_s^- = \{x \in G \mid \nu^N(x) \leq s\}$ are called the positive t-cut of B and the negative s-cut of B, respectively,*

(b) *the sets ${}^{>}B_t^+ = \{x \in G \mid \mu^P(x) > t\}$ and ${}^{<}B_s^- = \{x \in G \mid \nu^N(x) < s\}$ are called the strong positive t-cut of $B = (\mu^P, \nu^N)$ and the strong negative s-cut of $B = (\mu^P, \nu^N)$, respectively,*

(c) *the set $K_B^{(t,s)} = \{x \in G \mid \mu^P(x) \geq t,\ \nu^N(x) \leq s\}$ is called an (s, t)-level subset of B,*

(d) *the set ${}^{S}K_B^{(t,s)} = \{x \in G \mid \mu^P(x) > t,\ \nu^N(x) < s\}$ is called a strong (s, t)-level subset of B,*

(e) *the set of all $(s, t) \in Im(\mu^P) \times Im(\nu^N)$ is called the image of $B = (\mu^P, \nu^N)$.*

Theorem 5. *Let B be a bipolar fuzzy subquasigroup of $\mathcal{G}$ with the least upper bound (s_0, t_0). Then the following assertions are equivalent:*

(i) *B is a bipolar fuzzy subquasigroup of $\mathcal{G}$,*

(ii) *for all $(s, t) \in Im(\mu^P) \times Im(\nu^N)$, the nonempty level subset $K_B^{(s,t)}$ of B is a subquasigroup of $\mathcal{G}$.*

(iii) *for all $(s, t) \in Im(\mu^P) \times Im(\nu^N) \setminus (s_0, t_0)$, the nonempty strong level subset ${}^{S}K_B^{(s,t)}$ of B is a subquasigroup of $\mathcal{G}$.*

(iv) *for all $(s, t) \in [-1, 0] \times [0, 1]$, the nonempty strong level subset ${}^{S}K_B^{(s,t)}$ of B is a subquasigroup of $\mathcal{K}$.*

(v) *for all $(s, t) \in [-1, 0] \times [0, 1]$, the nonempty level subset $K_B^{(s,t)}$ of B is a subquasigroup of $\mathcal{G}$.*

Proof: (i $\to$ iv) : Let B be a bipolar fuzzy subquasigroup of $\mathcal{G}$, $(s, t) \in [-1, 0] \times [0, 1]$ and $x, y \in^{S} K_B^{(s,t)}$. Then we have

$$\mu^P(x * y) \geq \min\{\mu^P(x), \mu^P(y)\} > \min\{t, t\} = t$$

and

$$\nu^N(x * y) \leq \max\{\nu^N(x), \nu^N(y)\} < \max\{s, s\} = s.$$

Thus $x * y \in {}^{S}K_B^{(s,t)}$. Hence ${}^{S}K_B^{(s,t)}$ is a subquasigroup of $\mathcal{G}$.
(iv $\to$ iii) : The proof of this part is trivial.
(iii $\to$ ii) : Let $(s, t) \in Im(\mu^P) \times Im(\nu^N)$. Then $K_B^{(s,t)}$ is nonempty. Since $K_B^{(s,t)} = \bigcap\limits_{s > \beta, t < \alpha}^{S} K_B^{(s,t)}$, where $\beta \in Im(\mu^P) \setminus s_0$ and $\alpha \in Im(\nu^N) \setminus t_0$. Then by (iii), $K_B^{(s,t)}$ is a subquasigroup of $\mathcal{K}$.
(ii $\to$ v) : Let $(s, t) \in [-1, 0] \times [0, 1]$ and $K_B^{(s,t)}$ be

nonempty. Suppose that $x, y \in K_B^{(s,t)}$. Let $\alpha = \min\{\mu^P(x), \mu^P(y)\}$ and $\beta = \max\{\nu^N(x), \nu^N(y)\}$. Then, it is clear that $\alpha \geq s$ and $\beta \leq t$. Thus $x, y \in K_B^{(s,t)}$ and $\alpha \in Im(\mu^P)$ and $\beta \in Im(\nu^N)$, by (ii) we see that $K_B^{(\alpha,\beta)}$ is a subquasigroup of $\mathcal{G}$, and hence $x * y \in K_B^{(\alpha,\beta)}$. Thus, we have

$$\mu^P(x * y) \geq \{\mu^P(x), \mu^P(y)\} \geq \min\{\alpha, \alpha\} = \alpha \geq s$$

and

$$\nu^N(x * y) \leq \{\nu^N(x), \nu^N(y)\} \leq \max\{\beta, \beta\} = \beta \leq t.$$

Therefore $x * y \in K_B^{(s,t)}$. This proves that $K_B^{(s,t)}$ is a subquasigroup of $\mathcal{G}$.

(v $\rightarrow$ i) : Assume that the nonempty set $K_B^{(s,t)}$ is a subquasigroup of $\mathcal{G}$ for any $(s,t) \in [-1,0] \times [0,1]$. On the contrary, we assume that $x_0, y_0 \in G$ such that

$$\mu^P(x_0 * y_0) < \min\{\mu^P(x_0), \mu^P(y_0)\}$$

and

$$\nu^N(x_0 * y_0) > \max\{\nu^N(x_0), \nu^N(y_0)\}.$$

Let $\mu^P(x_0) = \alpha$, $\mu^P(y_0) = \beta$, $\mu^P(x_0 * y_0) = \lambda$, $\nu^N(x_0) = \theta$, $\nu^N(y_0) = \gamma$ and $\nu^N(x_0 * y_0) = \nu$. Then $\lambda < \min\{\alpha, \beta\}, \nu > \max\{\theta, \gamma\}$. Put

$$\lambda_1 = \frac{1}{2}(\mu^P(x_0 * y_0) + \min\{\mu^P(x_0), \mu^P(y_0)\})$$

and

$$\nu_1 = \frac{1}{2}(\nu^N(x_0 * y_0) + \max\{\nu^N(x_0), \nu^N(y_0)\}),$$

therefore $\lambda_1 = \frac{1}{2}(\lambda + \min\{\alpha, \beta\}), \nu_1 = \frac{1}{2}(\nu + \max\{\theta, \gamma\})$. Hence $\alpha > \lambda_1 = \frac{1}{2}(\lambda + \min\{\alpha, \beta\}) > \lambda, \nu > \nu_1 = \frac{1}{2}(\nu + \max\{\theta, \gamma\}) > \theta$. Thus $\min\{\alpha, \beta\} > \lambda_1 > \lambda = \mu^P(x_0 * y_0), \max\{\gamma, \theta\} < \nu_1 < \nu = \nu^N(x_0 * y_0)$, so that $x_0 * y_0 \notin K_B^{(\lambda_1, \nu_1)}$, a contradiction since $\mu^P(x_0) = \alpha \geq \min\{\alpha, \beta\} > \lambda_1 \mu^P(y_0) = \beta \geq \min\{\alpha, \beta\} > \lambda_1$, and $\nu^N(x_0) = \theta \leq \max\{\gamma, \theta\} < \nu_1, \nu^N(y_0) = \gamma \leq \max\{\gamma, \theta\} < \nu_1$ imply that $x_0, y_0 \in K_B^{(\lambda_1, \nu_1)}$. Thus

$$\mu^P(x * y) \geq \min\{\mu^P(x), \mu^P(y)\}$$

and

$$\nu^N(x * y) \leq \max\{\nu^N(x), \nu^N(y)\}$$

for all $x, y \in G$.

This completes the proof. □

Theorem 6. *Each subquasigroup of $\mathcal{G}$ is a level subquasigroup of a bipolar fuzzy subquasigroup of $\mathcal{G}$.*

Proof: Let Y be a subquasigroup of $\mathcal{G}$ and B a bipolar fuzzy subset of $\mathcal{G}$ defined by

$$\mu^P(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

$$\nu^N(x) = \begin{cases} \beta & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in [0,1]$ and $\beta \in [-1,0]$. It is clear that $K_B^{(t,s)} = Y$. Let $x, y \in G$. We consider the following cases :

Case 1: If $x, y \in Y$, then $x * y \in Y$ and therefore $\mu^P(x * y) = \alpha = \min\{\alpha, \alpha\} = \min\{\mu^P(x), \mu^P(y)\}$ and $\nu^N(x * y) = \beta = \max\{\beta, \beta\} = \max\{\nu^N(x), \nu^N(y)\}$.

Case 2: If $x, y \notin Y$, then $\mu^P(x) = 0 = \mu^P(y)$ and $\nu^N(x) = 0 = \nu^N(y)$ and so

$\mu^P(x * y) =\geq 0 = \min\{0,0\} = \min\{\mu^P(x), \mu^P(y)\}$ and $\nu^N(x * y) \leq 0 = \max\{0,0\} = \max\{\nu^N(x), \nu^N(y)\}$.

Case 3: If $x \in Y$ and $y \notin Y$, then $\mu^P(y) = 0 = \nu^N(y)$, $\mu^P(x) = \alpha$ and $\nu^N(x) = \beta$. Thus $\mu^P(x * y) \geq 0 = \min\{\mu^P(x), \mu^P(y)\}$ and $\nu^N(x * y) \leq 0 = \max\{\nu^N(x), \nu^N(y)\}$.

Case 4: If $x \notin Y$ and $y \in Y$, then by using the same argument as in Case 3, we can conclude the results. This proves that B is a bipolar fuzzy subquasigroup of $\mathcal{G}$. □

Theorem 7. *Let $\mathcal{G}$ be a quasigroup. Then for any chain of subquasigroups*

$$A_0 \subset A_1 \subset \cdots \subset A_r = \mathcal{G}$$

there exists a bipolar fuzzy subquasigroup B of $\mathcal{G}$ whose level subquasigroups are exactly the subquasigroups of this chain.

Proof: Consider set of numbers $p_0 > p_1 > \cdots > p_r$ and $q_0 < q_1 < \cdots < q_r$ where each $p_i \in [0,1]$ and $q_i \in [-1,0]$. Define μ^P and ν^N by $\mu^P(A_i \setminus A_{i-1}) = p_i$, for all $0 < i \leq r$, and $\mu^P(A_0) = p_0$, and $\nu^N(A_i \setminus A_{i-1}) = q_i$, for all $0 < i \leq r$, and $\nu^N(A_0) = q_0$. We prove that $B = (\mu^P, \nu^N)$ is a bipolar fuzzy subquasigroup of $\mathcal{G}$. Let $x, y \in \mathcal{G}$, we consider the following cases:

Case 1: If $x, y \in A_i \setminus A_{i-1}$, then $\mu^P(x) = p_i = \mu^P(y)$ and $\nu^N(x) = q_i = \nu^N(y)$. Since A_i is a subquasigroup thus $x * y \in A_i$, we have $\mu^P(x * y) \geq p_i = \min\{\mu^P(x), \mu^P(y)\}$, and $\nu^N(x * y) \leq q_i = \max\{\nu^N(x), \nu^N(y)\}$.

Case 2: If $x \in A_i \backslash A_{i-1}$ and $y \in A_j \backslash A_{j-1}$, where $i < j$. Then $\mu^P(x) = p_i$, $\mu^P(y) = p_j$, $\nu^N(x) = q_i$ and $\nu^N(y) = q_j$. Since $A_j \subset A_i$ and A_j is a subquasigroup of $\mathcal{G}$, then $x * y \in A_j$. Hence $\mu^P(x * y) \geq p_j = \min\{\mu^P(x), \mu^P(y)\}$, and $\nu^N(x * y) \leq q_j = \max\{\nu^N(x), \nu^N(y)\}$. It is clear that $Im(\mu^P) = \{p_0, p_1, \ldots, p_r\}$ and $Im(\nu^N) = \{q_0, q_1, \ldots, q_r\}$, therefore the level subquasigroups of μ^P and ν^N are given by the chain of subquasigroups

$$(\mu^P_{p_0}, \nu^N_{q_0}) \subset (\mu^P_{p_1}, \nu^N_{q_1}) \ldots \subset (\mu^P_{p_r}, \nu^N_{q_r}) = \mathcal{G}.$$

We have $(\mu^P_{p_0}, \nu^N_{q_0}) = \{x \in \mathcal{G} \mid \mu^P(x) \geq p_0, \nu^N(x) \leq q_0\} = A_0$. It is clear that $A_i \subseteq (\mu^P_{p_i}, \nu^N_{q_i})$. Let $x \in (\mu^P_{p_i}, \nu^N_{q_i})$ then $\mu^P(x) \geq p_i$ and $\nu^N(x) \leq q_i$ then $x \notin A_j$ for $j > i$. So $\mu^P(x) \in \{p_0, p_1, \ldots, p_i\}$ and $\nu^N(x) \in \{q_0, q_1, \ldots, q_i\}$, thus $x \in A_k$ for $k \leq i$, since $A_k \subseteq A_i$ we get that $x \in A_i$. Hence $A_i = (\mu^P_{p_i}, \nu^N_{q_i})$, for $0 \leq i \leq r$. □

Definition 8. *A quasigroup X is said to be Artinian if it satisfies the descending chain condition on subquasigroups of $\mathcal{G}$ (simply written as DCC), that is, for every chain $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ of subquasigroups of $\mathcal{G}$, there is a natural number i such that $I_i = I_{i+1} = \cdots$.*

Theorem 9. *Let $\mathcal{G}$ be a quasigroup. Then each bipolar fuzzy subquasigroup of $\mathcal{G}$ has finite values if and only if $\mathcal{G}$ is Artinian.*

Proof: Suppose that each bipolar fuzzy subquasigroup of $\mathcal{G}$ has finite values. If $\mathcal{G}$ is not Artinian, then there is a strictly descending chain

$$G = I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq I_n \supsetneqq \cdots$$

of subquasigroups of $\mathcal{G}$, where $I_i \supsetneqq I_j$ expresses $I_i \supseteq I_j$ but $I_i \neq I_j$. We now construct the bipolar fuzzy set $B = (\mu^P,\, \nu^N)$ of $\mathcal{K}$ by

$$\mu^P(x) := \begin{cases} \frac{n}{n+1} & \text{if} \quad x \in I_n \setminus I_{n+1},\ n = 1, 2, \cdots, \\ 1 & \text{if} \quad x \in \bigcap_{n=1}^{\infty} I_n, \end{cases}$$
$$\nu^N(x) := \ -\mu^P(x).$$

We first prove that B is a bipolar fuzzy subquasigroup of $\mathcal{K}$. For this purpose, we need to verify that μ^P is a fuzzy subquasigroup of $\mathcal{G}$. We assume that $x, y \in G$. Now, we consider the following cases:

Case 1: $x, y \in I_n \setminus I_{n+1}$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$\mu^P(x * y) \geq \frac{n}{n+1} = \min\{\mu^P(x),\, \mu^P(y)\}.$$

Case 2: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n < m)$. In this case, $x, y \in I_n$, and $x * y \in I_n$. Thus

$$\mu^P(x * y) \geq \frac{n}{n+1} = \min\{\mu^P(x),\, \mu^P(y)\}.$$

Case 3: $x \in I_n \setminus I_{n+1}$ and $y \in I_m \setminus I_{m+1} (n > m)$. In this case, $x, y \in I_m$, and $x * y \in I_m$. Thus

$$\mu^P(x * y) \geq \frac{m}{m+1} = \min\{\mu^P(x),\, \mu(y)\}.$$

Therefore μ^P satisfies (BF1), and so μ^P is a fuzzy subquasigroup of $\mathcal{G}$. This shows that B is a bipolar fuzzy subquasigroup of $\mathcal{G}$, but the values of B are infinite, a contradiction. Thus $\mathcal{G}$ is Artinian.

Conversely, suppose that $\mathcal{G}$ is Artinian. If there is a bipolar fuzzy subquasigroup $B = \{\mu^P,\, \nu^N\}$ of $\mathcal{G}$ with $|Im(B)| = +\infty$, then $|Im(\mu^P)| = +\infty$ or $|Im(\nu^N)| = +\infty$. Without loss of generality, we may assume that $Im(\mu^P) = +\infty$. Select $s_i \in Im(\mu^P)(i = 1, 2 \cdots)$ and $s_1 < s_2 < \cdots$. Then $U(\mu^P;\, s_i)(i = 1,\, 2,\, \cdots)$ are subquasigroups of $\mathcal{G}$ and $U(\mu;\, s_1) \supseteq U(\mu^P;\, s_2) \supseteq \cdots$ with $U(\mu^P;\, s_i) \neq U(\mu^P;\, s_{i+1})\,(i = 1,\, 2,\, \cdots)$, a contradiction. Similar for $\mathrm{Im}(\nu^N)$. The proof is completed. □

Definition 10. *A quasigroup $\mathcal{G}$ is said to be* Noetherian *if every subquasigroup of $\mathcal{G}$ is finitely generated. $\mathcal{G}$ is said to satisfy* the ascending chain condition (briefly, ACC) *if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of subquasigroups of $\mathcal{G}$ there is a natural number n such that $I_i = I_n$ for all $i \geq n$.*

Theorem 11. *$\mathcal{G}$ is Noetherian if and only if for any bipolar fuzzy subquasigroup A, the set $Im(B)$ is a well ordered subset, that is, $(Im(\mu^P),\, \leq)$ and $(Im(\nu^N),\, \geq)$ are well ordered subsets of $[0,\, 1]$, respectively.*

Proof: $(\Rightarrow)$ Suppose that $\mathcal{G}$ is Noetherian. For any chain $t_1 > t_2 > \cdots$ of $Im(\mu^P)$, let $t_0 = \inf\{t_i | \, i = 1,\, 2,\, \cdots\}$. Then $I := \{x \in G | \, \mu^P(x) > t_0\}$ is a subquasigroup of $\mathcal{G}$, and so I is finitely generated. Let $I = (a_1,\, \cdots,\, a_k]$. Then $\mu^P(a_1) \wedge \cdots \wedge \mu^P(a_k)$ is the least element of the chain $t_1 > t_2 > \cdots$. Thus $(Im(\mu^P),\, \leq)$ is a well ordered subset of $[0,\, 1]$. By using the same argument as above, we can easily show that $(Im(\nu^N),\, \geq)$ is a well ordered subset of $[0,\, 1]$. Therefore, $Im(B)$ is a well ordered subset.

$(\Leftarrow)$Let $Im(B)$ be a well ordered subset. If $\mathcal{G}$ is not Noetherian, then There is a strictly ascending sequence of subquasigroups of $\mathcal{K}$ such that

$$I_1 \subset I_2 \subset \cdots.$$

We construct the bipolar fuzzy set $B = (\mu^P,\, \nu^N)$ of $\mathcal{G}$ by

$$\mu^P(x) := \begin{cases} \frac{1}{n} & \text{if} \quad x \in I_n - I_{n-1},\ n = 1, 2, \cdots, \\ 0 & \text{if} \quad x \notin \bigcup_{n=1}^{\infty} I_n, \end{cases}$$
$$\nu^N(x) := \ -\mu^P(x)$$

where $I_0 = .$ By using similar method as the necessity part of Theorem 3.15, we can prove that B is a bipolar fuzzy subquasigroup of $\mathcal{G}$. Because $Im(B)$ is not well ordered, which is a contradiction. This completes the proof. □

ACKNOWLEDGMENT

The author thanks the Dean of Scientific Research of King Abdulaziz University for the support to this paper under grant no. 1431/363/170.

REFERENCES

1. Zadeh, L.A., 1965. Fuzzy sets, Inform. Control, 8:338-353.
2. Zhang, W.-R., 1994. Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis, Proc. of IEEE Conf., 305-309.
3. Zhang, W.-R., 1998. Bipolar fuzzy sets, Proc. of FUZZ-IEEE, 835-840.
4. Lee, K.-M. Lee, 2004. Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets, J. Fuzzy Logic Intell. Sys., 14:125-129.
5. Dubois, D. and Kaci, S. and Prade, H., 2004. Bipolarity in Reasoning and Decision, an Introduction, Int. Con. on Inf. Pro. Man. Unc. IPMU'04,. 959-966.
6. Akram, M., and Dudek, W.A., 2008. Generalized fuzzy subquasigroups, Quasigroups and Related Systems, 16:133-146.
7. Akram, M., and Dudek, W.A., 2009. New fuzzy subquasigroups, Quasigroups and Related Systems, 17:
8. Akram, M., 2008. Fuzzy subquasigroups with respect to a s-norm, Bul. Acad. Sci. Republ. Moldova, ser. Mathematica, 2:3-13.
9. Akram, M., Saeid, A. B., Shum, K. P. and Meng, B. L., 2010. Bipolar fuzzy K-algebras, International Journal of Fuzzy System, 10:252-258.
10. Akram M. and Dar K. H., 2010. Generalized fuzzy K-algebras, VDM Verlag, ISBN 978-3-639-27095-2
11. Bloch, I., 2007. Mathematical morphology on bipolar fuzzy sets, Pro. 8th Inter. Sym. Math. Morp., 2:3-4.
12. Bloch, I., 2007. Dilation and erosion of spatial bipolar fuzzy sets, Lecture Notes in Artificial Intelligence,. 385-393.
13. Dudek, W. A., 1998. Fuzzy subquasigroups, Quasigroups and Related Systems, 5:81-98.
14. Dudek, W. A., 2001. On some old and new problems in n-ary groups, Quasigroups and Related Systems, 8:15-36.
15. Dudek, W.A. and Jun, Y.B., 1999. Fuzzy subquasigroups over a t-norm, Quasigroups and Related Systems, 6:87-98.
16. Lee, K.-M., 2009. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malay. Math. Sci. Soc., 32:361-373.
17. Lee, K.-M., 2000. Bip olar-valued fuzzy sets and their basic operations, Proc. Int. Conf., Bangkok, Thailand, 307-317.
18. Rosenberg, I.G., 2000. Two properties of fuzzy subquasigroups of a quasigroup, Fuzzy Sets and Systems, 110:447-450.
19. Rosenfeld, A., 1971. Fuzzy groups, J. Math. Anal. Appl., 35:512-517.
20. Jun, Y.B., Park, C. H., 200. Filters of BCH-algebras based on bipolar-valued fuzzy sets, International Mathematical Forum, 13:631-643.