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Analytic Solutions of Some Partial Differential Equations by Using Homotopy Perturbation Method

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Abstract: In this paper, we use the homotopy perturbation method (HPM) in order to find the analytic solutions of Clannish random walkers equation, three-dimensional nonlinear parabolic partial differential equation, molecular recombination problem and a inhomogeneous coupled Burger's equation. The method does not need linearization or weak nonlinearity assumptions. In this scheme, solution is calculated in the form of a convergent power series with easily computable components.

Key words: Homotopy perturbation method (HPM) · Clannish random walkers equation · Three-dimensional linear parabolic equation · Molecular recombination problem · Inhomogeneous coupled Burger's equation

INTRODUCTION

The purpose of the present letter is to find the analytic solution of some partial differential equations by using HPM [1-3] presented by Dr. He in 1998. In this paper, it is considered homogeneous and inhomogeneous various multidimensional linear and nonlinear parabolic partial differential equations. The method is handled more easily, quickly and elegantly without linearizing the problem by implementing the homotopy perturbation method [2, 4, 5] rather than the traditional methods for finding the analytic solutions. In this study, we will not use any transformation to reduce the multidimensional nonlinear problem to a system of simpler partial differential equations or any linearization. The original nonlinear equation is directly solvable preserving the actual physics and involving much less calculation. This scheme will be illustrated by studying various multidimensional linear and nonlinear equations to compute approximate solution. Furthermore, we will also illustrate self-cancelling phenomena for inhomogeneous coupled Burger's equation using the homotopy perturbation method. The method is useful for obtaining both approximate and numerical approximations of linear or nonlinear differential equations. In the literature, this method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations [6-8]. Some scientists have studied on analytical solution of linear or nonlinear differential equations with different methods [9-11].

An Analysis of the Method and its Applications: In this section, homotopy perturbation method is considered. In stead of other perturbation methods, this method doesn't need a small parameter in an equation. According to this method, a homotopy with an imbedding parameter $p\in[0,1]$ is constructed and the imbedding parameter is considered as a "small parameter". Thus, this method is called homotopy perturbation method. To illustrate this method, we consider the following nonlinear differential equation,

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{2.1}$$

With boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,\ r \in \Gamma, \tag{2.2}$$

Where $A(u)$ is written as follows:

$$A(u) = L(u) + N(u). \tag{2.3}$$

A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the

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boundary of the domain Ω. The operator A can be generally divided into two parts L and N, where L linear operator is and N is nonlinear operator. Thus, Eq. (2.1) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0. \tag{2.4}$$

By the homotopy technique, it is constructed a homotopy $v(r,p)$: $\Omega \times [0,1] \rightarrow \square$ which satisfies

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, p \in [0,1], r \in \Omega \tag{2.5}$$

Where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (2.1), which satisfies the boundary conditions. Clearly, from Eq. (2.5) we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{2.6}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{2.7}$$

the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic.
We consider v as following:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + ... = \sum_{n=0}^{\infty} p^n v_n \tag{2.8}$$

According to homotopy perturbation method, the best approximation solution of Eq. (2.4) can be explain as a series of the power of p,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + ... = \sum_{n=0}^{\infty} v_n \tag{2.9}$$

Convergence of the series (2.8) is given in [1, 2]. Some results have been discussed in [12-15].

Example 1: We consider a clannish random walkers equation [16] in order to illustrate the method for nonlinear problem. The problem is given by

$$u_t - u_{xx} - \alpha\left(u^2\right)_x + \alpha u_x = 0 \tag{2.10}$$

Where we choose $\alpha = 1$, with initial condition $u(x,0) = \frac{1}{x}$. For (2.10) equation we can construct a homotopy as following:

$$(1-p)\left[\dot{Y} - \dot{u}_0\right] + p\left[\dot{Y} - Y'' - \left(Y^2\right)' + Y'\right] = 0 \tag{2.11}$$

Where $\dot{Y} = \frac{\partial Y}{\partial t}$, $Y' = \frac{\partial Y}{\partial x}$, $Y'' = \frac{\partial^2 Y}{\partial x^2}$ and $p \in [0,1]$.

With initial approximation $Y_0 = u_0 = \frac{1}{x}$, suppose the solution of Eq. (2.11) has the form:

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + ... \tag{2.12}$$

Then, substituting Eq. (2.12) into Eq. (2.11) and equating the terms with same powers of p,

$$p^0 : \frac{\partial Y_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \tag{2.13}$$

$$p^1 : \frac{\partial Y_1}{\partial t} - \frac{\partial^2 Y_0}{\partial x^2} - \frac{\partial\left(Y_0^2\right)}{\partial x} + \frac{\partial Y_0}{\partial x} = 0, \tag{2.14}$$

$$p^2 : \frac{\partial Y_2}{\partial t} - \frac{\partial^2 Y_1}{\partial x^2} - 2\frac{\partial\left(Y_0 Y_1\right)}{\partial x} + \frac{\partial Y_1}{\partial x} = 0, \tag{2.15}$$

$$p^3 : \frac{\partial Y_3}{\partial t} - \frac{\partial^2 Y_2}{\partial x^2} - 2\frac{\partial\left(Y_0 Y_2\right)}{\partial x} - \frac{\partial\left(Y_1^2\right)}{\partial x} + \frac{\partial Y_2}{\partial x} = 0, \tag{2.16}$$

$\vdots$

With solving Eqs. (2.13)- (2.16)

$$Y_0 = \frac{1}{x},\ Y_1 = \frac{t}{x^2},\ Y_2 = \frac{t^2}{x^3},\ Y_3 = \frac{t^3}{x^4},... \tag{2.17}$$

Thus, as considering Eq. (2.12) with (2.17) and suppose $p = 1$, we obtain analytic solution of Eq. (2.10) as following:

$$u = \frac{1}{x} + \frac{t}{x^2} + \frac{t^2}{x^3} + \frac{t^3}{x^4} + ... = \frac{1}{x-t} \tag{2.18}$$

Example 2: For comparison purposes, we consider three-dimensional linear parabolic equation [16] with initial condition

$$u_t - u_{xx} + u_{yy} + u = (1+t)\sinh(x+y),\ u(x,y,0) = \sinh(x+y). \tag{2.19}$$

In order to solve this equation by using the homotopy perturbation method, we can construct a homotopy as following:

$$(1-p)\left[\dot{Y}-\dot{u}_0\right]+p\left(\frac{\partial Y}{\partial t}-\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial y^2}+Y-(1+t)\sinh(x+y)\right)=0 \quad (2.20)$$

With initial approximation $Y_0 = u_0 = \sinh(x + y)$, suppose the solution of Eq. (2.20) has the form:

$$Y = Y_0 + pY_1 + p^2Y_2 + p^3Y_3 + \cdots. \quad (2.21)$$

Then, substituting Eq. (2.21) into Eq. (2.20) and equating the terms with same powers of p,

$$p^0 : \frac{\partial Y_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad (2.22)$$

$$p^1 : \frac{\partial Y_1}{\partial t} + \frac{\partial u_0}{\partial t} - \frac{\partial^2 Y_0}{\partial x^2} + \frac{\partial^2 Y_0}{\partial y^2} - (1+t)Sinh(x+y) = 0, \quad (2.23)$$

$$p^2 : \frac{\partial Y_2}{\partial t} - \frac{\partial^2 Y_1}{\partial x^2} + \frac{\partial^2 Y_1}{\partial y^2} + Y_1 = 0, \quad (2.24)$$

$$p^3 : \frac{\partial Y_3}{\partial t} - \frac{\partial^2 Y_2}{\partial x^2} + \frac{\partial^2 Y_2}{\partial y^2} + Y_2 = 0, \quad (2.25)$$

$\vdots$

With solving Eqs. (2.22)- (2.25)

$$Y_0 = \sinh(x+y),\ Y_1 = \frac{t^2}{2!}\sinh(x+y),$$
$$Y_2 = -\frac{t^3}{3!}\sinh(x+y),\ Y_3 = \frac{t^4}{4!}\sinh(x+y), \quad (2.26)$$

Thus, as considering Eq. (2.21) with (2.26) and suppose $p = 1$, we obtain analytic solution of Eq. (2.19) as following:

$$u(x,y,t) = \left(e^{-t} + t\right)Sinh(x+y). \quad (2.27)$$

Example 3: In the case of nonlinear equation, we first consider a molecular recombination problem [16] in order to illustrate the method for multidimensional nonlinear equation. The equation is given with the initial condition by

$$u_t - u_{xx} + u_{yy} + u^2 = 0,\ u(x,y,0) = \frac{1}{x-y}. \quad (2.28)$$

Taking the equation (2.19), we can construct a homotopy as following:

$$(1-p)\left[\dot{Y}-\dot{u}_0\right]+p\left(\frac{\partial Y}{\partial t}-\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial y^2}+Y^2\right)=0 \quad (2.29)$$

With initial approximation $Y_0 = u_0 = \frac{1}{x-y}$, suppose the solution of Eq. (2.29) has the form Eq. (2.21) and as same Example 1, substituting Eq. (2.21) into Eq. (2.29) and equating the terms with same powers of p,

$$p^0 : \frac{\partial Y_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, \quad (2.30)$$

$$p^1 : \frac{\partial Y_1}{\partial t} - \frac{\partial^2 Y_0}{\partial x^2} + \frac{\partial^2 Y_0}{\partial y^2} + Y_0^2 = 0 \quad (2.31)$$

$$p^2 : \frac{\partial Y_2}{\partial t} - \frac{\partial^2 Y_1}{\partial x^2} + \frac{\partial^2 Y_1}{\partial y^2} + 2Y_0Y_1 = 0 \quad (2.32)$$

$\vdots$

With solving Eqs. (2.30)- (2.32)

$$Y_0 = \frac{1}{x-y},\ Y_1 = -\frac{t}{(x-y)^2},\ Y_2 = \frac{t^2}{(x-y)^3},\ldots \quad (2.33)$$

Thus, as considering Eq. (2.21) with (2.33) and suppose $p = 1$, we obtain analytic solution of Eq. (2.28) as following:

$$u(x,y,t) = \frac{1}{x-y+t}. \quad (2.34)$$

Example 4: As an example of the application of the self-canceling phenomena, we seek the analytic solution of the inhomogeneous coupled Burger's equation [16].

$$u_t - u_{xx} + uu_x + (uv)_x = x^2 - 2t + 2x^3t^2 + t^2,$$
$$v_t - v_{xx} + vv_x + (uv)_x = \frac{1}{x} - \frac{2t}{x^3} - \frac{t^2}{x^3} + t^2. \quad (2.35)$$

Subject to the initial conditions

$$u(x,0) = 0, \quad v(x,0) = 0 \quad (2.36)$$

For the solution of these equations, we simply construct a homotopy as following:

$$(1-p)\left(\dot{Y}-\dot{u}_0\right)+p\left(\dot{Y}-Y''+YY'+(YV)'-x^2+2t-2x^3t^2-t^2\right)$$
$$(1-p)\left(\dot{V}-\dot{v}_0\right)+p\left(\dot{V}-V''+VV'+(YV)'-\frac{1}{x}+\frac{2t}{x^3}+\frac{t^2}{x^3}-t^2\right) \quad (2.37)$$

Where $\dot{Y}=\frac{\partial Y}{\partial t}$, $Y'=\frac{\partial Y}{\partial x}$, $Y''=\frac{\partial^2 Y}{\partial x^2}$, $\dot{V}=\frac{\partial V}{\partial t}$, $V'=\frac{\partial V}{\partial x}$, $V''=\frac{\partial^2 V}{\partial x^2}$ and $p\in[0,1]$ suppose the solution of Eqs. (2.37) have the form:

$$Y=Y_0+pY_1+p^2Y_2+p^3Y_3+\ldots=\sum_{n=0}^{\infty}Y_n(x,t)$$
$$V=V_0+pV_1+p^2V_2+p^3V_3+\ldots=\sum_{n=0}^{\infty}V_n(x,t) \quad (2.38)$$

substituting Eqs. (2.38) into Eq. (2.37) and equating the terms with same powers of p,

$$p^0:\dot{Y}_0-\dot{u}_0=0,$$
$$p^1:\dot{Y}_1+\dot{u}_0-Y_0''+Y_0Y_0'+(Y_0V_0)'-x^2+2t-2x^3t^2-t^2=0,$$
$$p^2:\dot{Y}_2-Y_1''+Y_0Y_1'+Y_1Y_0'+(Y_0V_1)'+(Y_1V_0)'=0, \quad (2.39)$$
$$p^3:\dot{Y}_3-Y_2''+Y_0Y_2'+Y_1Y_1'+Y_2Y_0'+(Y_0V_2)'+(Y_1V_1)'+(Y_2V_0)'=0,$$
$$\vdots$$

and

$$p^0:\dot{V}_0-\dot{v}_0=0,$$
$$p^1:\dot{V}_1+\dot{v}_0-V_0''+V_0V_0'+(Y_0V_0)'-\frac{1}{x}+\frac{2t}{x^3}+\frac{t^2}{x^3}-t^2=0,$$
$$p^2:\dot{V}_2-V_1''+V_0V_1'+V_1V_0'+(Y_0V_1)'+(Y_1V_0)'=0, \quad (2.40)$$
$$p^3:\dot{V}_3-V_2''+V_0V_2'+V_1V_1'+V_2V_0'+(Y_0V_2)'+(Y_1V_1)'+(Y_2V_0)'=0$$

With solving Eqs. (2.39)- (2.40)

$$Y_1=x^2t-t^2+\frac{2x^3t^3}{3}+\frac{t^3}{3},$$
$$Y_2=t^2-xt^4, \quad (2.41)$$
$$Y_3=-\frac{t^3}{3}+\frac{t^4x}{2}-\frac{2t^3x^3}{3}+\frac{3t^5}{5x^4}-\frac{t^4}{2x^2}+\ldots$$

$$V_1=\frac{t}{x}-\frac{t^2}{x^3}-\frac{t^3}{3x^3}+\frac{t^3}{3}$$
$$V_2=\frac{t^2}{x^3}-\frac{t^4}{x^5}-\frac{4t^3}{x^5}, \quad (2.42)$$
$$V_3=-\frac{t^3}{3}+\frac{4t^3}{x^5}+\frac{t^3}{3x^3}-\frac{t^4}{2x^2}+\frac{3t^5}{5x^4}-\frac{8t^5}{15}+\ldots$$
$$\vdots$$

It is obvious that the self-canceling "noise" terms [17] appear between various components, looking into the second, third and fourth terms of Y_1 and V_1 (2.41), (2.42). Keeping the remaining non-canceled terms and using (2.38) leads immediately to the solutions of (2.35) with initial conditions (2.36) given by

$$u(x,t)=x^2t,\quad v(x,t)=\frac{t}{x}. \quad (2.43)$$

CONCLUSION

Nonlinear phenomena play a crucial role in applied mathematics and physics. Furthermore, when an original nonlinear equation is directly calculated, the solution will preserve the actual physical characters of solutions. Therefore the explicit solutions of the nonlinear equations have a fundamental importance. Various effective methods have been developed to understand the mechanisms of physical models, to help physicist and engineers and to ensure knowledge for physical problems and its applications.

And the other the advantage is that the approximation of the series methodologies can display a fast convergence of the solutions. The illustrations show the dependence of the rapid convergence depends on the character and behavior of the solutions just as in a closed form solutions. We point out that, for given equations with initial values $u(x,0)$, the corresponding analytical and numerical solutions are obtained according to the recurrence relations using Mathematica package version of Mathematica 4 in PC computer.

REFERENCES

1. He, J.H., 1999. Homotopy perturbation technique. Computer Methods in Applied Mechanics and Engineering, 178: 257-262.
2. He, J.H., 2000. A coupling method of homotopy technique and a perturbation technique for nonlinear problems. International J. Non-Linear Mechanics, 35: 37-43.
3. He, J.H., 2003. Homotopy perturbation method a new nonlinear analytical technique. Applied Mathematics and Computation, 135: 73-79.
4. He, J.H., 1998. An approximate solution technique depending on an artificial parameter: A special example. Communications in Nonlinear Science and Numerical Simulation, 3: 92-97.
5. He, J.H., 2004. Asymptotology by homotopy perturbation method. Applied Mathematics and Computation, 156: 591-596.
6. Yıldırım, A. and A. Kelleci, 2009. Numerical simulation of the Jaulent-Miodek equation by He's homotopy perturbation method. World Applied Sciences Journal, 7: 84-89.
7. Oziş, T. and A. Yıldırım, 2008. Comparison between Adomian's method and He's homotopy perturbation method. Computers and Mathematics with Applications, 56: 1216-1224.
8. Yıldırım, A., 2009. Application of He s homotopy perturbation method for solving the Cauchy reaction-diffusion problem. Computers and Mathematics with Applications, 57: 612-618.
9. Ma, W.X. and J.H. Lee, 2009. A transformed rational function method and exact solutions to the 3 + 1 dimensional Jimbo-Miwa equation. Chaos, Solitons and Fractals, 42: 1356-1363.
10. Ma, W.X. and Y. You, 2004. Rational solutions of the Toda lattice equation in Casoratian form. Chaos, Solitons and Fractals, 22: 395-406.
11. Ma, W.X. and C.X. Li and J. He, 2009. A second Wronskian formulation of the Boussinesq equation. Nonlinear Analysis, 70: 4245-4258.
12. He, J.H., 2004. The homotopy perturbation method for nonlinear oscillators with discontinuities. Applied Mathematics and Computation, 151: 287-292.
13. Inc, M. and Y. Ugurlu, 2007. Numerical simulation of the regularized long wave equation by He's homotopy perturbation method. Physics Letters A., 369: 173-179.
14. Oziş, T. and A. Yıldırım, 2008. Reliable analysis for obtaining exact soliton solutions of nonlinear Schrödinger (NLS) equation. Chaos, Solitons and Fractals, 38: 209-212.
15. Sadighi, A. and D.D. Ganji, 2007. Exact solutions of Laplace equation by homotopy-perturbation and Adomian decomposition methods. Physics Letters A., 367: 83-87.
16. Ames, W.F., 1965. Nonlinear Partial Differential Equations in Engineering. Academic Press, New York.
17. Adomian, G. and R. Rach, 1992. Noise terms in decomposition series solution. Computers and Mathematics with Applications, 24: 61-24.