World Applied Sciences Journal 12 (1): 21-27, 2011
ISSN 1818-4952

A Taylor Series Based Method for Solving Nonlinear Sine-Gordon and Klein-Gordon Equations

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Abstract: In this paper, Taylor series based on which is the differential transform method (DTM) is applied to solve various forms of nonlinear Klein-Gordon type equations. The application of differential transform method is extended to derive approximate analytical solutions of nonlinear Klein-Gordon type equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components. Some examples are solved as illustrations, using symbolic computation. The results show that the approach is easy to implement and accurate when applied to these type equations. The method introduces a promising tool for solving many linear and nonlinear differential equations.

Key words: Two dimensional differential transform method · Sine-Gordon and klein-Gordon equations

INTRODUCTION

It is well known that many phenomena in scientific fields can be described by nonlinear partial differential equations. The nonlinear models of real-life problems are still difficult to solve either numerically or analytically. A broad class of analytical solution methods and numerical methods were used to handle these problems. For some nonlinear problems, although exact analytical solutions can be achieved, they often appear in terms of very complicated implicit functions and are not convenient for application.

For example, the nonlinear Klein-Gordon equations arise in a variety of physical circumstances and the initial-value problem of the one-dimensional nonlinear Klein-Gordon equation is given by the following form.

$$u_{tt} + \alpha u_{xx} + g(u) = f(x,t) \qquad (1)$$

Where:
$u = u(x,t)$ stands for the wave displacement at position x and time t, α is a known constant and $g(u)$ is a nonlinear force. For example, in well-known sine-Gordon equation, the nonlinear force is given by $g(u) = sin\ u$. In different physical applications, the nonlinear force $g(u)$ has also other forms, such as Klein-Gordon Equation with a Power Law Nonlinearity [1]. Some numerical methods for solving Eq.(1) are given in [1-5] and the references therein.

Several techniques based on series expansion, Adomian decomposition method [6-8], Variational iteration method [9-11] and Homotopy perturbation method [12], Homotopy analysis method [13] and auxiliary equation method [14] has been used for the solution of these equations.

In this paper, Taylor's series based method so called the differential transform method (DTM) is applied to solve various forms of nonlinear Klein-Gordon type equations. This method is different from the traditional high order Taylor's series method which requires symbolic computation of the necessary derivatives of the data function. Traditional Taylor series method is computationally taken long time for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The solutions of our model equations are calculated in the form of convergent series with easily computable components by using symbolic computation which is in suitable form for application.

Differential Transform Method: Differential transform method (DTM) first introduced by Zhou [15] and its main

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application therein was to solve both linear and nonlinear initial value problems arising in electrical circuit theory. Differential transform method (DTM) is a semi-analytical numerical technique that basically uses Taylor series for the solution of differential equations in the form of a polynomial with a distinct algorithm. Consequently, the DTM is an alternative choice for getting Taylor series solution of the given differential equations. There are also other methods in the literature based on Taylor series expansion such as Restricted Taylor series method [3] and Adomian decomposition method [6-8]. But differential transform method (DTM) formulizes the Taylor series in a totally different way. With this method, the given differential equation and related boundary conditions are transformed into a recurrence relations which leads to the solution of a system of algebraic equations as coefficients of a power series solution and is easily carried out in computer. Because of this property, the method is no need of linearization of the nonlinear problems and as a result avoids the large computational works and the round-off errors. Differential Transform method has been successfully applied to various problems [16-24] recently. In this section we briefly describe differential transform method as follow:

One-Dimensional Differential Transform

Definition: If $u(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time t,

$$\frac{\partial^k u(t)}{\partial t^k} = \phi(t,k) \text{ for all } t \in T \tag{2}$$

for $t = t$, then $\phi(t,k) = \phi(t_i,k)$, where k belongs to the set of nonnegative integers, denoted as the K-domain. Therefore, Eq. (2) can be rewritten as

$$U(k) = \phi(t_i,k) = \left[\frac{\partial^k u(t)}{\partial t^k}\right]\Big|_{t=t_i} \tag{3}$$

Where:

$U(k)$ is called the spectrum of $u(t)$ at $t = t_i$.

Definition 2.2. If $u(t)$ can be expressed by Taylor's series, then $u(t)$ can be represented as

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]U(k) \tag{4}$$

Eq.(4) is called the inverse of $u(t)$, with the symbol D denoting the differential transformation process. Combining (3) and (4), we obtain

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]U(k) \equiv D^{-1}U(k) \tag{5}$$

Applying the differential transformation, a differential equation in the domain of interest can be transformed to algebraic equation in the K-domain and the $u(t)$ can be obtained by finite-term Taylor's series plus the remainder, as.

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]U(k) + R_{n+1}(t) \tag{6}$$

In order to speed up the convergent rate and the accuracy of calculation, the entire domain of t needs to be split into sub-domains [2].

Two-Dimensional Differential Transform: Let $w(x,y)$ a function of two variables such that $w(x,y)$ is an analytic function in the domain K and let $(x,y) = (x_0, y_0)$ in this domain. The function $w(x,y)$ is then represented by a power series whose center located at (x_0, y_0). The differential transform of function $w(x,y)$ is.

$$W(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} \tag{7}$$

Where:

$w(x,y)$ is the original function and $W(k,h)$ is the transformed function. The differential inverse transform of $W(k,h)$ is defined as follows:

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)(x-x_0)^k (y-y_0)^h \tag{8}$$

Combining Eqs.(7) and (8), it can be obtained that

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h} w(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)}(x-x_0)^k (y-y_0)^h \tag{9}$$

From the above definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion with Eqs. (7) and (8), the fundamental mathematical operations performed by two-dimensional differential transform can readily be obtained and are listed in Table 1.

Applications: To illustrate the effectiveness of the present method, several test examples are considered in this section.

Table 1: Operations for the two-dimensional differential transformation Original function Transformed function

$w(x,y)=u(x,y)\pm v(x,y)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(x,y)=\alpha u(x,y)$	$W(k,h)=\alpha U(k,h)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial x}$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial y}$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,y)=\dfrac{\partial u^{r+s}(x,y)}{\partial x^r \partial y^s}$	$W(k,h)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)U(k+r,h+s)$
$w(x,y)=u(x,y)v(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,y)=x^m y^n$	$W(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)=\begin{cases}1, & for\ k=m \quad and\ h=n\\ 0, & otherwise\end{cases}$
$w(x,y)=u(x,y)v(x,y)\dfrac{\partial c^2(x,y)}{\partial x^2}$	$W(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(k-r-t+2)(k-r-t+1)U(r,h-s-p)V(t,s)C(k-r-t+2,p)$
$w(x,y)=\dfrac{\partial u(x,y)}{\partial x}\dfrac{\partial v(x,y)}{\partial x}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)V(k-r+1,s)$
$w(x,y)=u(x,y)\dfrac{\partial v^2(x,y)}{\partial x^2}$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+2)(k-r+1)U(r,h-s)V(k-r+2,s)$
$w(x,y)=u(x,y)v(x,y)q(x,y)$	$W(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}U(r,h-s-p)V(t,s)Q(k-r-t,p)$

Example 1. We first consider the sine-Gordon equation

$$u_u - u_{xx} = \sin u \tag{10}$$

subject to initial conditions

$$u(x,0)=\frac{\pi}{2}, \qquad u_t(x,0)=0 \tag{11}$$

Taking in to consideration $\sin u = u-\frac{u^3}{3!}+\frac{u^5}{5!}-...$, then the transformed version of Eq.(10) is

$$\begin{aligned}&(h+1)(h+2)U(k,h+2)-(k+1)(k+2)U(k+2,h)=\\ &U(k,h)-\frac{1}{6}\left(\sum_{r=0}^{k}\left(\sum_{t=0}^{k-r}\left(\sum_{s=0}^{h}\left(\sum_{p=0}^{h-s}U(r,h-s-p)U(t,s)U(k-r-t,p)\right)\right)\right)\right)\end{aligned} \tag{12}$$

The transformed initial conditions are

$$U(k,0)=\frac{\pi}{2} \qquad k=0,1,2,... \tag{13}$$

and

$$U(k,1)=0 \tag{14}$$

respectively. Substituting (13) and (14) into (12), we obtained the closed form solution as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k t^h$$

$$\begin{aligned}&=0.5000000000\pi+0.4624161147t^2-0.009005575050t^4\\ &\quad-0.005527867713t^6-0.0001544012065t^8+0.00005500492296t^{10}\end{aligned}$$

or

$$=\frac{\pi}{2}+\frac{1}{2}t^2-\frac{1}{240}t^6+\frac{1}{172800}t^{10}+... \tag{15}$$

which is the solution of the problem.

Example 2: Consider the sine-Gordon equation

$$u_u - u_{xx} + \sin u = 0 \tag{16}$$

subject to initial conditions

$$u(x,0) = 0,\ u_t(x,0) = 4 \sec h(x) \tag{17}$$

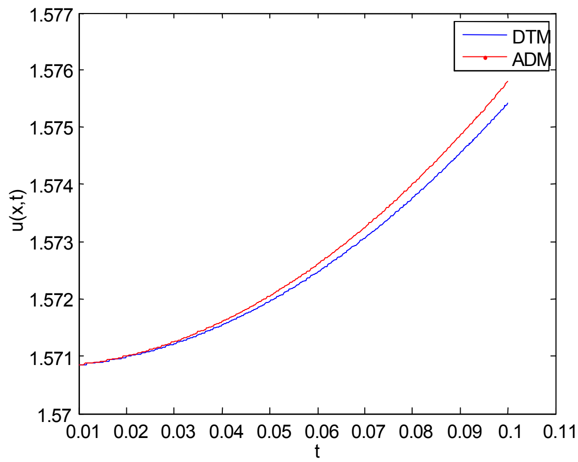

Fig. 1: The approximate solution of sine-gordon equation by using ADM and DTM

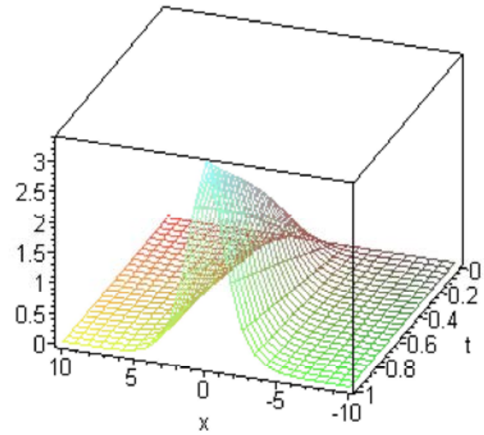

Fig. 2: Comparison between exact solution and differential transform method for $u(x,t)$.

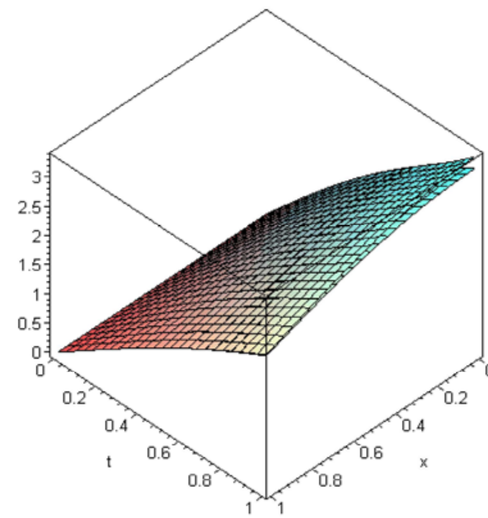

Fig. 3: Comparison between exact solution and differential transform method for $x = 0,..1$ and $t = 0,..,1$

Table 2: Comparison of the present method with ADM [40] for various t values

t	DTM	ADM
0.01	1.570842569	1.570846327
0.02	1.570981292	1.570996327
0.03	1.571212494	1.571246327
0.04	1.571536170	1.571596327
0.05	1.571952311	1.572046327
0.06	1.572460908	1.572596327
0.07	1.573061949	1.573246327
0.08	1.573755420	1.573996326
0.09	1.574541304	1.574846325
0.1	1.575419582	1.575796323

Taking into consideration $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - ...$, then the transformed version of Eq.(16) is

$$(h+1)(h+2)U(k,h+2)-(k+1)(k+2)U(k+2,h)= U(k,h)-\frac{1}{6}\left(\sum_{r=0}^{k}\left(\sum_{t=0}^{k-r}\left(\sum_{s=0}^{h}\left(\sum_{p=0}^{h-s}U(r,h-s-p)U(t,s)U(k-r-t,p)\right)\right)\right)\right) \tag{18}$$

The transformed initial conditions are

$$U(k,0) = 0 \qquad k = 0,1,2,... \tag{19}$$

$$U(k,1) = 4\sec h(x) \qquad k = 0,1,2,... \tag{20}$$

respectively. Substituting (19) and (20) into (18), we obtain the closed form solution as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k t^h$$

$$= 4t - \frac{4}{3}t^3 + \frac{4}{5}t^5 - \frac{116}{315}t^7 - 2x^2t + 2x^2t^3 - 2x^2t^5 + \frac{97}{63}x^2t^7 + \frac{5}{6}x^4t - \frac{11}{6}x^4t^3 + \frac{17}{6}x^4t^5 - \frac{1177}{378}x^4t^7 - \frac{61}{180}x^6t + \frac{241}{180}x^6t^3 - \frac{541}{180}x^6t^5 + \frac{51887}{11340}x^6t^7 + \frac{277}{2016}x^8t - \frac{8651}{10080}x^8t^3 + \frac{26837}{10080}x^8t^5 - \frac{3412537}{635040}x^8t^7 ...$$

or

$$= \left(4 - 2x^2 + \frac{5}{6}x^4 - \frac{61}{180}x^6 + \frac{277}{2016}x^8\right)t + \left(-\frac{4}{3} + 2x^2 - \frac{11}{6}x^4 + \frac{241}{180}x^6 - \frac{8651}{10080}x^8\right)t^3 + \left(\frac{4}{5} - 2x^2 + \frac{17}{6}x^4 - \frac{541}{180}x^6 + \frac{26837}{10080}x^8\right)t^5 + \left(-\frac{116}{315} + \frac{94}{63}x^2 - \frac{1177}{378}x^4 + \frac{51887}{11340}x^6 - \frac{3412537}{635040}x^8\right)t^7 + ... \tag{21}$$

hich gives the closed form solution by [9].

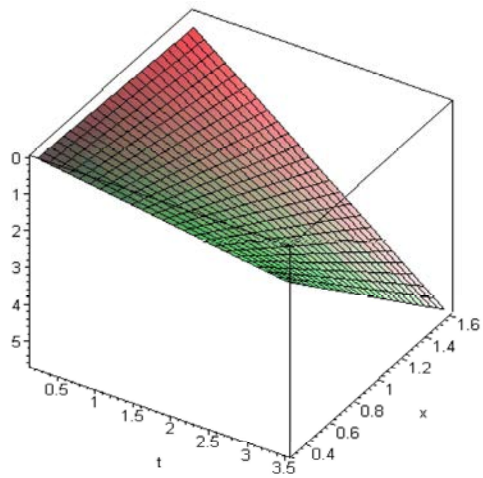

Fig. 4: Solution of klein-Gordon equation for x = 0.3,...,1.6 and t = 0.2,..., 3.5.

$$v(x,t) = 4 \arctan (t \sec h(x)) \tag{22}$$

Example 3: Consider the nonlinear non-homogeneous Klein-Gordon equation

$$u_u - u_{xx} = u^2 = x^2t^2 \tag{23}$$

subject to initial conditions

$$u(x,0) = 0 \ , \ u_t(x,0) = x \tag{24}$$

The transformed version of Eq.(23) is

$$\begin{aligned}&(h+1)(h+2)U(k,h+2)-(k+1)(k+2)U(k+2,h)\\&+\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)U(k-r,s)=\delta(k-2)\delta(h-2)\end{aligned} \tag{25}$$

The transformed initial conditions are

$$U(k,0) = 0 \tag{26}$$

and

$$U(k,1)=\begin{cases}1, & k=1\\ 0, & otherwise\end{cases} \tag{27}$$

respectively.

Substituting (26) and (27) in (25), we obtained the solution as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$u(x,t) = xt \tag{28}$$

which is the exact solution of (23).

Table 3: Omparison of the differential transform method and exact solution for values of (x,t)

(x,t)	DTM	Exact solution	Absolute Error
(0,0)	0	0	0
(0.1,0,1)	0.3967025516	0.3967025318	$0.198.10^{-7}$
(0.2,0.2)	0.7744407152	0.7744383605	0.2354710^{-7}
(0.3,0.3)	1.117938909	1.117903310	0.000035599
(0.4,0.4)	1.417703903	1.417478390	0.000225513
(0.5,0.5)	1.670046492	1.669172441	0.000874051
(0.6,0.6)	1.875503673	1.873003332	0.002500341
(0.7,0.7)	2.034159224	2.027881163	0.006278061
(0.8,0.8)	2.119859965	2.101787350	0.018072615
(0.9,0.9)	1.935206961	1.863811562	0.071395399
(1,1)	0.4479024943	0.1202380952	0.3276643991

Table 4: Comparison between exact solution for x=0.01 and 5-iterative MADM, 2-iterative VIM, 3-iterative HPM and 4-iterative DTM

t	|*Exact-MADM*|	|*Exact-VIM*|	|*Exact-HPM*|	|*Exact-DTM*|
0.01	1.32E-06	5.00E-07	6.00E-16	6.56E-25
0.02	1.05E-05	4.00E-06	8.11E-14	1.34E-21
0.03	3.49E-05	1.35E-05	1.38E-12	1.16E-19
0.04	8.19E-05	3.19E-05	1.04E-12	2.75E-18
0.05	1.58E-04	6.23E-05	4.92E-11	3.21E-17
0.06	2.71E-04	1.07E-04	1.76E-10	2.38E-16
0.07	4.25E-04	1.70E-04	5.16E-10	1.30E-15
0.08	6.28E-04	2.54E-04	1.31E-09	5.64E-15
0.09	8.84E-04	3.60E-04	2.97E-09	2.06E-14
0.1	1.20E-03	4.92E-04	6.18E-09	6.56E-14

Table 5: Comparison between exact solution for x=0.01 and 5-iterative MADM, 2-iterative VIM, 3-iterative HPM and 4-iterative DTM.

t	|*Exact-MADM*|	|*Exact-VIM*|	|*Exact-HPM*|	|*Exact-DTM*|
0.01	1.93E-04	4.97E-07	5.00E-16	6.28E-25
0.02	3.93E-04	4.00E-06	7.33E-14	1.29E-21
0.03	6.08E-04	1.34E-05	1.25E-12	1.11E-19
0.04	8.45E-04	3.18E-05	9.36E-12	2.63E-18
0.05	1.11E-03	6.20E-05	4.45E-11	3.07E-17
0.06	1.41E-03	1.07E-04	1.59E-10	2.28E-16
0.07	1.76E-03	1.69E-04	4.66E-10	1.24E-15
0.08	2.15E-03	2.52E-04	1.18E-09	5.39E-15
0.09	2.59E-03	3.58E-04	2.68E-09	1.97E-14
0.1	3.09E-03	4.90E-04	5.58E-09	6.28E-14

CONCLUSION

In this work, differential transform method is extended to solve the nonlinear Klein-Gordon equations. The present study has confirmed that the differential transform method offers significant advantages in terms of its straightforward applicability, its computational effectiveness and its accuracy.

ACKNOWLEDGEMENTS

The authors thank the referees for their valuable suggestions.

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