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Focal Curve of Timelike Biharmonic Curve in the Lorentzian Heisenberg Group $Heis^3$

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Abstract: In this paper, we study focal curve of timelike biharmonic curve in the Heisenberg group $Heis^3$. We characterize focal curve of timelike biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group $Heis^3$ Finally, we construct parametric equations of focal curve of timelike biharmonic curve.

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INTRODUCTION

Darboux had found how to determine the evolutes of a curve, that is, the curves whose tangents are normals of γ. Moreover, he had shown that the focal surface of γ is foliated by the evolutes and all of them lie on the focal surface.

The differential geometry of space curves is a classical subject which usually relates geometrical intuition with analysis and topology. For any unit speed curve γ, the focal curve $C\gamma$ is defined as the centers of the osculating spheres of γ. Since the centerof any sphere tangent to at a point lies on the normal plane to γ at that point, the focal curve of γ may be parameterized using the Frenet frame $(\mathrm{t}(s),\mathrm{n}_1(s),\mathrm{n}_2(s))$ of γ as follows:

$$C_\gamma(s) = (\gamma + c_1\mathrm{n}_1 + c_2\mathrm{n}_2)(s),$$

Where the coefficients c_1, c_2 are smooth functions that are called focal curvatures of γ.

The aim of this paper is to study focal curve of a timelike biharmonic curve in the Lorentzian Heisenberg group $Heis^3$.

Harmonic maps $f:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \mathrm{trace}\nabla df \tag{1.2}$$

The bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

The first and the second variation formula for the bienergy, showing that the Euler-Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathrm{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df,\tau(f))df = 0, \tag{1.4}$$

Where J^f is the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study focal curve of timelike biharmonic curve in the Heisenberg group $Heis^3$. We characterize focal curve of timelike biharmonic curve in terms of curvature and torsion of biharmonic curve in the Heisenberg group $Heis^3$ Finally, we construct parametric equations of focal curve of timelike biharmonic curve.

The Lorentzian Heisenberg Group $Heis^3$: The Lorentzian Heisenberg group $Heis^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

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$(\bar{x},\bar{y},\bar{z})(x,y,z) = (\bar{x}+x,\bar{y}+y,\bar{z}+z-\bar{x}y+x\bar{y}).$

$Heis^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.
The Lorentz metric g is given by

$$g = dx^2 + dy^2 + (xdy + dz)^2.$$

The Lie algebra of $Heis^3$ has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$e_2,e_3] = 2e_1,[e_3,e_1] = 0, [e_2,e_1] = 0$$

With

$$g(e_1,e_1) = g(e_2,e_2) = 1, g(e_3,e_3) = -1.$$

Proposition 2.1: For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

Where the (i,j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k = 1,2,3\} = \{e_1,e_2,e_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = -g(R(X,Y)Z,W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a,\mathbf{e}_b)\mathbf{e}_c, R_{abcd} = R(\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c,\mathbf{e}_d),$$

Where the indices a,b,c and d take the values 1,2 and 3.

$R_{232} = -3R_{131} = -3e_3,$
$R_{133} = -R_{122} = e_1,$
$R_{233} = -3R_{121} = -3e_2,$
and

$$R_{1212} = -1, R_{1313} = 1, R_{2323} = -3. \tag{2.3}$$

Timelike Biharmonic Curves in the Lorentzian Heisenberg Group $Heis^3$: Let $\gamma: I \to Heis^3$ be a timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. Let $\{t,n_1,n_2\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $Heis^3$ along γ defined as follows:

t is the unit vector field γ' tangent to γ, n_1 is the unit vector field in the direction of $\nabla_t t$ (normal to γ) and n_2 is chosen so that $\{t,n_1,n_2\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_t t &= \kappa n_1, \\ \nabla_t n_1 &= \kappa t + \tau n_2, \\ \nabla_t n_2 &= -\tau n_1, \end{aligned} \tag{3.1}$$

Where κ is the curvature of γ and τ is its torsion.
With respect to the orthonormal basis $\{e_1,e_2,e_3\}$ we can write

$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3,$$

$$\mathbf{n}_1 = n_1^1\mathbf{e}_1 + n_1^2\mathbf{e}_2 + n_1^3\mathbf{e}_3,$$

$$\mathbf{n}_2 = \mathbf{t}\times\mathbf{n}_1 = n_2^1\mathbf{e}_1 + n_2^2\mathbf{e}_2 + n_2^3\mathbf{e}_3.$$

Theorem 3.1: (see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a timelike non-geodesic biharmonic curve if and only if

$$\kappa = \text{constant} \neq 0,$$

$$\kappa^2 - \tau^2 = -1+4\,(n_2^1)^2 \tag{3.2}$$

$$\tau' = -2\, n_1^1\, n_2^1.$$

Corollary 3.2: (see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if

$$\begin{aligned}&\kappa = \text{constant} \neq 0,\\ &\tau = \text{constant},\\ &n_1^1\, n_2^1 = 0,\\ &\kappa^2 - \tau^2 = -1 + 4(n_2^1)^2.\end{aligned} \tag{3.3}$$

Theorem 3.3.: (see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $n_1^{\ 1} \neq 0$, then γ is not biharmonic.

Theorem 3.4.: (see [18]) Let $\gamma : I \to Heis^3$ be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 = 0$, then

$$\mathbf{t}(s) = \sinh\phi_0 \mathbf{e}_1 + \cosh\phi_0 \sinh\psi(s)\mathbf{e}_2 + \cosh\phi_0 \cosh\psi(s)\mathbf{e}_3, \tag{3.4}$$

Where $\phi_0 \in$ R.

Focal Curve of Timelike Biharmonic Curve in the Lorentzian Heisenberg Group Heis³: For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write

$$C_\gamma(s) = (\gamma + c_1 \mathrm{n} + c_2 \mathrm{b})(s), \tag{4.1}$$

Where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, c_2 = \frac{c_1'}{\tau}, \kappa \neq 0, \tau \neq 0. \tag{4.2}$$

Lemma 4.1.: Let $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve and and C_γ its focal curve on $Heis^3$. Then,

$$c_1 = \frac{1}{\kappa} = \text{constant} \text{and} c_2 = 0. \tag{4.3}$$

Proof: Using (3.3) and (4.2), we get (4.3).
Lemma 4.2. Let $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve and C_γ its focal curve on $Heis^3$. Then,

$$C_\gamma(s) = (\gamma + c_1 \mathrm{n})(s). \tag{4.4}$$

Theorem 4.3.: Let $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve and and C_γ its focal curve on $Heis^3$. Then, the parametric equations of C_γ are

$$\begin{aligned}x_{C_\gamma}(s) &= \frac{c_1}{\kappa}(\Re\cosh\phi_0 \sinh[\Re s + \varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \varepsilon])\\ &+ \frac{1}{\Re}\cosh\phi_0 \sinh[\Re s + \varepsilon] + \varepsilon_1,\\ y_{C_\gamma}(s) &= \frac{c_1}{\kappa}(\Re\cosh\phi_0 \cosh[\Re s + \varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \varepsilon])\\ &+ \frac{1}{\Re}\cosh\phi_0 \sinh[\Re s + \rho] + \varepsilon_2,\\ z_{C_\gamma}(s) &= \frac{c_1}{\kappa}(\Re\cosh\phi_0 \cosh[\Re s + \varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \varepsilon])\\ &(\frac{\kappa}{\Re}\cosh\phi_0 \sinh[\Re s + \varepsilon] + \frac{2\kappa}{\Re^2}\sinh\phi_0 \cosh\phi_0 \cosh[\Re s + \varepsilon] + \varepsilon_3 s + \varepsilon_4)\\ &+ \frac{1}{\Re}\cosh\phi_0 \sinh[\Re s + \varepsilon] - \frac{1}{\Re}\cosh^2\phi_0 (-\frac{s}{2} + \frac{\sinh 2[\Re s + \varepsilon]}{4\Re})\\ &- \frac{\varepsilon_1}{\Re}\cosh\phi_0 \cosh[\Re s + \varepsilon] + \varepsilon_5,\end{aligned} \tag{4.5}$$

Where ε, ε_1, ε_2, ε_3, ε_4, ε_5, are constants of integration and $\Re = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

Proof: The covariant derivative of the vector field t is:

$$\nabla_\mathbf{t}\mathbf{t} = t_1'\mathbf{e}_1 + (t_2' + 2t_1 t_3)\mathbf{e}_2 + (T_3' + 2t_1 t_2)\mathbf{e}_3. \tag{4.6}$$

From (3.4), we have

$$\begin{aligned}\nabla_\mathbf{t}\mathbf{t} &= (\psi'\cosh\phi_0 \cosh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))\mathbf{e}_2\\ &+ (\psi'\cosh\phi \sinh\psi(s) + 2\sinh\phi_0 \cosh\phi_0 \cosh\psi(s))\mathbf{e}_3.\end{aligned} \tag{47}$$

Since $|\nabla_t t| = \kappa$ we obtain

$$\psi(s) = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)s + \varepsilon, \tag{4.8}$$

Where $\varepsilon \in$ R.

Thus (3.4) and (4.8), imply

$$\mathbf{t} = \sinh\phi_0 e_1 + \cosh\phi_0 \sinh[\Re s + \varepsilon]e_2 + \cosh\phi_0 \cosh[\Re s + \varepsilon]e_3, \tag{4.9}$$

Where $\Re = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

Using (3.1) in (4.9), we obtain

$$\mathbf{t} = (\cosh\phi_0 \cosh[\Re s+\varepsilon], \cosh\phi_0 \sinh[\Re s+\varepsilon], \cosh\phi_0 \cosh[\Re s+\varepsilon]) \\ -x(s)\cosh\phi_0 \sinh[\Re s+\varepsilon]).$$

From (2.1), we get

$$\mathbf{t} = (\cosh\phi_0 \cosh[\Re s+\varepsilon], \cosh\phi_0 \sinh[\Re s+\varepsilon], \cosh\phi_0 \cosh[\Re s+\varepsilon]) \\ -\left(\frac{1}{\Re}\cosh\phi_0 \sinh[\Re s+\varepsilon]+\varepsilon_1\right)\cosh\phi_0 \sinh[\Re s+\varepsilon]), \qquad (4.10)$$

Where ε_1 is constant of integration.

From (3.1) and (4.9), we get

$$\nabla_{\mathbf{t}}\mathbf{t} = (\Re\cosh\phi_0 \cosh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon])\mathbf{e}_2$$

$$+(\Re\cosh\phi_0 \sinh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon])\mathbf{e}_3.$$

Where $\Re = (\pm\dfrac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

By the use of Frenet formulas, we get

$$\mathbf{n}_1 = \frac{1}{\kappa}\nabla_{\mathbf{t}}\mathbf{t}$$
$$= \frac{1}{\kappa}[(\Re\cosh\phi_0 \cosh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon])\mathbf{e}_2$$

$$+(\Re\cosh\phi_0 \sinh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon])\mathbf{e}_3]. \qquad (4.11)$$

Substituting (2.1) in (4.11), we have

$$\mathbf{n}_1 = \frac{1}{\kappa}((\Re\cosh\phi_0 \sinh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon]),$$
$$(\Re\cosh\phi_0 \cosh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon]), \qquad (4.12)$$
$$(\Re\cosh\phi_0 \cosh[\Re s+\varepsilon] + 2\sinh\phi_0 \cosh\phi_0 \cosh[\Re s+\varepsilon])$$

$$.(\frac{\kappa}{\Re}\cosh\phi_0 \sinh[\Re s+\varepsilon] + \frac{2\kappa}{\Re^2}\sinh\phi_0 \cosh\phi_0 \\ \cosh[\Re s+\varepsilon] + \varepsilon_3 s + \varepsilon_4)),$$

Where ε_3, ε_4 are constants of integration.

Next, we substitute (4.10) and (4.12) into (4.4), we get (4.5). The proof is completed.

Using Mathematica in Theorem 4.3, yields

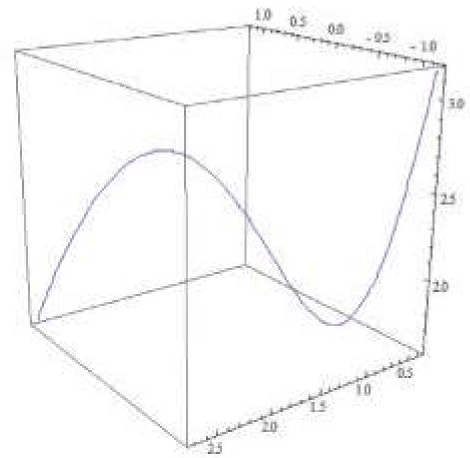

Fig. 1:

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