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Impulsive Differential Equations by Using the Second-Order Taylor Series Method

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Abstract: Impulsive differential equation is emerging as an important area of investigation since such equations appear to represent a natural framework for mathematical modeling of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory especially on the qualitative behavior of solutions of the impulsive differential equations with fixed moments. However, many impulsive differential equations cannot be solved analytically or their solving is complicated. In this paper, the algorithm which follows the theory of impulsive differential equations to solve the impulsive differential equations by using the second-order Taylor series methods is presented. Finally, the better convergence result of the numerical solution is given by solving the numerical examples.

Key words: Differential Equations · Impulsive Differential Equations · Fixed impulse · Impulsive jump · Numerical Method

INTRODUCTION

Impulsive differential equations, by means, differential equations involving impulse effects, are seen as a natural description of observed evolution phenomenon of several real world problems. For example, mechanical system with impact, biological phenomenon involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and industrial robotics and a host of others, do exhibit impulsive effects. Furthermore, impulses do contribute to the stabilization of some delay differential systems [1]. Therefore, it is beneficial to study the theory of impulsive differential equations as a well deserved discipline, due to the increase applications of impulsive differential equations in various fields in the future. The pioneer papers in this theory are written by Mil'man and Myshkis in 1960's [2].

In spite of its importance, many solutions regarded to impulsive differential equations are done analytically. Some of the famous researchers who presented significance results are Lakshmikantham, Bainov, Simeonov and many others [3-10]. However, many impulsive differential equations cannot be solved analytically or if done, their solving is very much complicated [11]. Therefore, numerical solutions of impulsive differential equations has to be studied and the results has to be improved. In this paper, the numerical solutions of impulsive differential equations are sought by using the Second-order Taylor Series method. The algorithm proposed is interpreted according to the theory of impulsive differential equations written by Lakshmikantham *et. al* [9]. The solutions are then compared to the previous article written by the researchers using the Euler method [12].

Preliminaries: Basically, impulsive differential equations consist of three components.

A continuous-time differential equation, which governs the state of the system between impulses, an impulse equation, which models an impulsive jump, defined by a jump function at the instant an impulse occurs and a jump criterion, which defines a set of jump events. Mathematically, the equation takes the form,

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$$x'(t) = f(t,x), \quad t \neq t_k, \quad t \in Z$$
$$\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, ..., m \tag{2.1}$$

where Z is any real interval, $f: Z \times R^n \to R^n$ is a given function, I_k: $R^n \to R^n$ $k = 1, 2, ..., m$ and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $k = 1, 2, ..., m$. The numbers t_k are called instants (or moments) of impulse, I_k represents the jump of state at each t_k, whereas $x(t_k^+)$ and $x(t_k^-)$ represent the right limit and the left limit, respectively, of the state at $t = t_k$. The moments of impulse maybe fixed or depended on the state of the system. In this paper, we will be concerned with fixed moments only.

The equations of systems with impulse at fixed moments have the following form

$$x'(t) = f(t,x), \quad t \neq t_k,$$
$$\Delta x = I_k(x), \quad t = t_k \tag{2.2}$$

where $t_0 < t_1 < ....t_k < t_{k+1} <....$, $k \in Z$ and for $t = t_k$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$.

We surely see that any solution, $x(t)$ of (2.2) satisfies

(i) $x'(t) = f(t, x(t)), \quad t \in (t_k, t_{k+1})$ and

(ii) $\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, ...$

Let the sets M(t) = M, N(t) = N and the operator A(t) = A be independent of t and let A:M→N be defined by Ax = $x + I(x)$, where I: Ω→Ω. Whenever, any solution $x(t) = x(t, 0, x_0)$ hits the set M at some time t, the operator A instantly transfers the point $x(t) = M$ into the point $y(t) + I(x(t)) \in N$.

Generally, the solutions of the impulsive differential equations are piecewise continuous functions with points of discontinuity at the moments of the impulse effect.

In this paper, we denote $S = \{t_k : k \in Z\} \subset R$ where $t_k < t_{k+1}$ for all $k \in Z$, $t_k \to +\infty$ when k → + ∞ and $k \to +\infty$ when $t_k \to -\infty$. If $\Omega \subset R$ is any real interval, we suppose that $x(t) = [x_1(t)\ x_2(t) .... x_n(t)]^T$, is a vector of unknown functions and

$$f: \Omega \times R^n \to R^n,$$

$$f(t,x) = \begin{bmatrix} f_1(t, x_1(t), x_2(t), ..., x_n(t) \\ f_2(t, x_1(t), x_2(t), ..., x_n(t) \\ ... \quad ... \quad ... \\ ... \quad ... \quad ... \\ f_n(t, x_1(t), x_2(t), ..., x_n(t) \end{bmatrix}$$

is continuous function on every set $[t_k, t_{k+1}] \times R$.

Definition 1: A system of differential equation in the form of

$$\frac{dx}{dt} = f(t,x), \quad t \neq t_k \tag{2.3}$$

with conditions

$$\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-) = I_k(x(t_k))$$

where I_k: $R^n \to R^n$ are continuous operators, $k = 0, \pm 1, \pm 2, ...$, is called impulsive differential equation (IDE) at fixed impulse. A state of the process,

$$x_0 = x(t_0)$$

is taken as the start condition to solve (2.3).

The problem of existence and uniqueness of the solutions of impulsive differential equations is similar to that of corresponding ordinary differential equations. The continuability of solutions is affected by the nature of the impulsive action.

Definition 2: A solution of the IDE (2.3) means a piecewise continuous $x: J \to R$ with piecewise continuous first derivative such that

1. $$\frac{dx(t)}{dt} = f(t, x(t)), \quad t \neq \tau_k$$
2. $$x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k)), \quad k = 0, \pm 1, \pm 2, ...$$

Let $x(t)$ be the solution of IDE (2.3) with initial condition $x(t)$, then $x(t)$ can be represented as

$$x(t) = \begin{cases} x_0 + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{t_0 < t_k < t} I_k(x(t_k)) & , \quad t \in \Omega^+ \\ x_0 + \int_{t_0}^{t} f(s, x(s))\, ds - \sum_{t_0 < t_k < t} I_k(x(t_k)) & , \quad t \in \Omega^- \end{cases}$$

where Ω^+ and Ω^- are the maximal intervals on which the solution can be continued to the right or to the left of the point $t = t_0$ respectively.

We also need the following known [9] impulsive differential inequalities result. For this purpose, we let PC denote the class of piecewise continuous functions from R_+ to R. with discontinuous of the first kind only at $t = t_k$; $k = 1,2....$, We can now state the needed results.

Theorem 2.1: Assumes that

(A_0) the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < ...t_k <...$ with $t_k \to \infty$ as $k \to \infty$;

(A_1) m $\in PC^1[R_+, R]$ and $m(t)$ is left continuous at t_k, k = 1,2,..,

(A_2) $\forall$k = 1,2... and $t \ge t_0$

$$\begin{cases} D^+m(t) \le g(t, m(t)), & t \ne t_k, \\ m(t_k^+) \le \psi_k(m(t_k)), & \\ m(t_0) \le w_0 & \end{cases} \tag{2.4}$$

where $g: R_+^2 \to R$ is continuous in $(t_{k-1}, t_k] \times R_+$ and for each $w \in R_+$

$\lim_{(t,z)\to(t_k^+,w)} g(t,z) = g(t_k^+, w)$ exists and $\Psi_k : R_+ \to R$ is non-decreasing;

(A_3) $r(t) = r(t, t_0, w_0)$ is the maximal solution of

$$\begin{cases} w' = g(t,w), & t \ne t_k \\ w(t_k^+) = \psi_k(w(t_k)), & \\ w(t_0) = w_0 \ge 0 & \end{cases} \tag{2.5}$$

existing on $[t_0, \infty]$. Then

$$m(t) \le r(t), \ t \ge t_0 \tag{2.6}$$

We recall that the maximal solution $r(t)$of (3.2) means the following

$$r(t) = \begin{cases} r_0(t, t_0, w_0), & t \in [t_0, t_1] \\ r_1(t, t_1, r_0(t_1^+)), & t \in (t_1, t_2] \\ \vdots & \\ r_k(t, t_k, r_{k-1}(t_k^+)) & t \in (t_k, t_{k+1}] \\ \vdots & \\ \vdots & \end{cases} \tag{2.7}$$

where each $r_i(t, t_i, r_{i-1}(t_i^+))$ is the maximal solution of (2.3) on the interval $(t_i, t_{i+1}]$ for each I = 1,2,...., and $r_{i-1}(t_i^+) = \psi_i(r_{i-1}(t_i, t_{i-1}, r_{i-2}(t_{i-1}^+)))$.

Numerical Examples: Suppose the IDE (2.3) with the start condition $x_0 = x(t_0)$ and the impulsive operators I_k, ($k \in Z$) are given. The impulsive operators act at the moments of jump happen, t_k for all $k \in Z$ which are described by the quadrate matrices of dimensions $n \times n$ The numerical algorithm in [12] is considered.

Example 1 Consider the IDE given in [13] **:**

$$\frac{dx}{dt} = f(t, x), \quad t \ne t_k$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, ... \tag{3.1}$$

$x_0 = x(t_0)$

and t_0 = 0.0,

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$f(t, x) = \begin{bmatrix} 0.1666666x_1 + 0.1666666x_2 + 0.1666666 \\ -0.1666666x_1 - 0.1666666x_2 + 0.5833333 \end{bmatrix}$$

The impulsive operators act at t_1 = 1.0 and t_2 = 2.0 are given as follows:

$$I_1 = \begin{bmatrix} 0.25 & 0.25 \\ 0.0 & -1.0 \end{bmatrix}, \qquad I_2 = \begin{bmatrix} 3.0 & 4.0 \\ 0.0 & -1.0 \end{bmatrix} \tag{3.2}$$

Here,we wish to approximate the value of t_k = 2.3.

We apply the algorithm by using the Second-order Taylor Series method,

$$x_{i+1} = x_i + hx_i' + \frac{h^2}{2!}x_i'' \tag{3.3}$$

where $i \in Z$ is the index of iteration and h is the step size of each iteration. Here, the step size h = 0.1. Then we compare the results obtained by using the analytical expression that is the solution of IDE (3.1).

The numerical values of solution are obtained by using the Matlab programming and the results of the Second-order Taylor Series method as well as the analytical expression are graphed in Figure 1 and 2.

Example 2: Consider the IDE,

$$\frac{dx}{dt} = f(t, x), \quad t \ne t_k$$

$$\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, ... \tag{3.4}$$

$x_0 = x(t_0)$

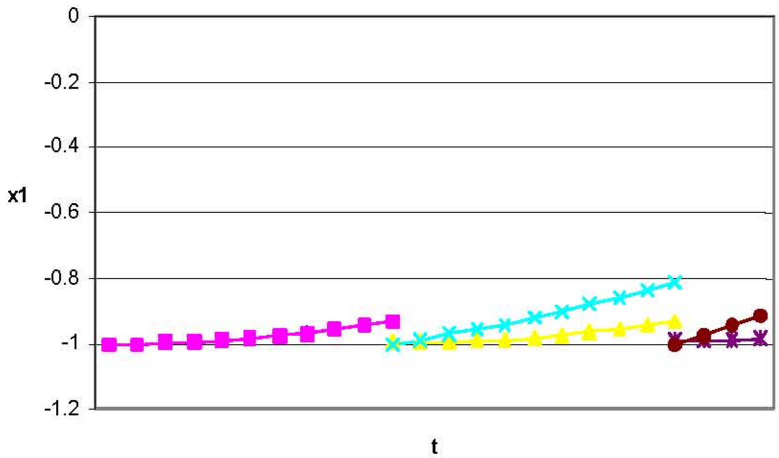

Fig. 1: The approximate values of x_1 between the Taylor and analytical method for Example 1

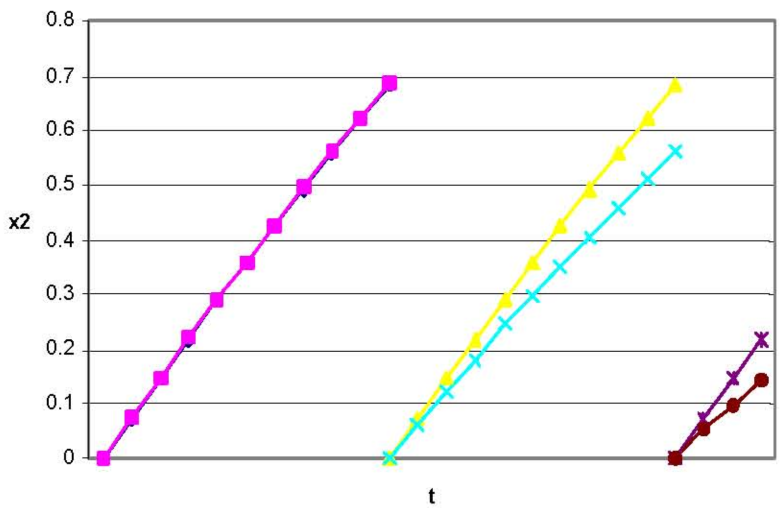

Fig. 2: The approximate values of x_2 versus time, t between the Taylor and analytical method for Example 1

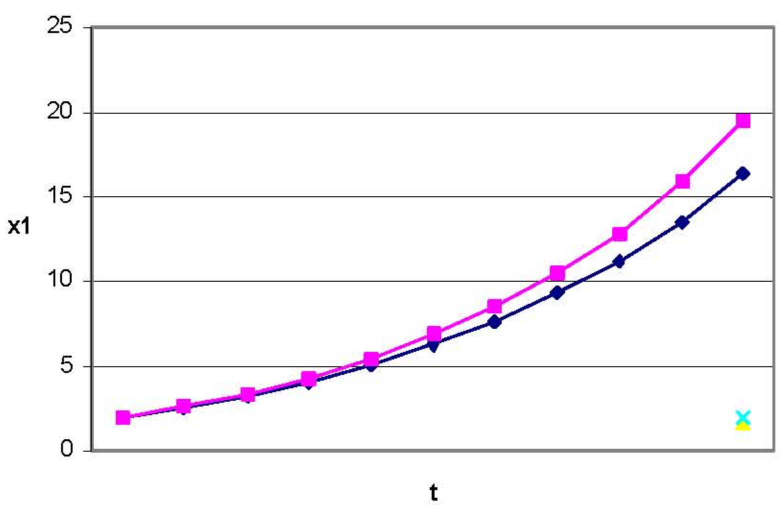

Fig. 3: The approximate values of x_1 versus time, t between the Taylor and analytical method for Example 2

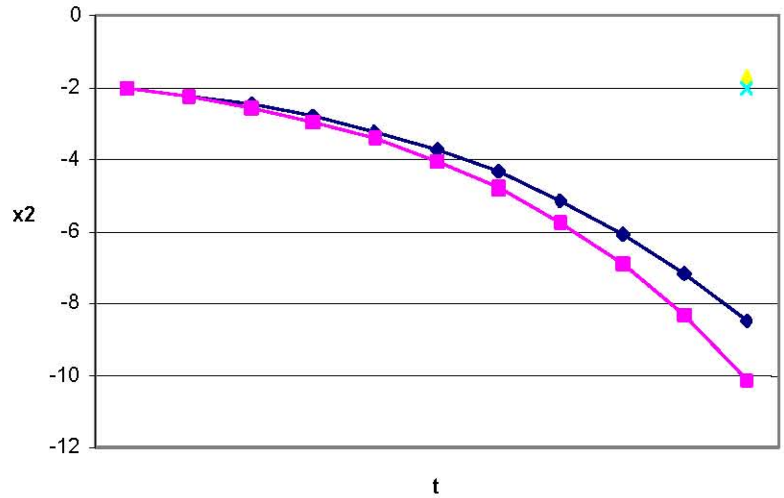

Fig. 4: The approximate values of x_2 versus time, t between the Taylor and analytical method for Example 2

Table 1: The comparison of errors of Example 1 between numerical methods at t = 2.3

t_k	Taylor $x_1(t_k)$	Euler $x_1(t_k)$	Taylor $x_2(t_k)$	Euler $x_2(t_k)$
2.3	0.0657	0.1032	0.0739	0.0781

Table 2: The comparison of errors of Example 2 between numerical methods at t = 1.0

t_k	Taylor $x_1(t_k)$	Euler $x_1(t_k)$	Taylor $x_2(t_k)$	Euler $x_2(t_k)$
1.0	0.3991	0.4761	0.3122	0.3187

and $t_0 = 0.0$,

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$f(t, x) = \begin{bmatrix} x_1 - 2x_2 \\ -x_1 \end{bmatrix}$$

At $k = 1$, $I_k = \begin{bmatrix} 1.0000 & 3.6565 \\ 0.0000 & -0.8020 \end{bmatrix}$. Here, we wish to determine the approximate value at $t_k = 1.0$.

For that purpose we apply the Second-order Taylor series method (3.3). Then we compare the results obtained by using the analytical expression that is the solution of IDE (3.4). The solutions are graphed in Figure 3 and 4.

CONCLUDING REMARKS

The accuracy of the results can be improved by investigating the solutions of the other numerical methods. In this paper, a general numerical procedure for treating the impulsive differential equations at fixed moments is proposed. The numerical algorithm is developed accordingly to the theory of impulsive differential equations for the second-order Taylor series method. Despite of its importance in the applications of real life problems, solving the impulsive differential equations numerically has not been done by many researchers. Therefore, many studies have to be done in order to enhance and verify the existing results. In this paper, we have shown that the second-order Taylor series method has improved the accuracy of the solutions.

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