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Timelike Involute Curve of Spacelike Biharmonic General Helices with Timelike Normal in the Lorentzian Group of Rigid Motions E(1,1)

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Abstract: In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions E(1,1). We characterize the timelike involute of spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions E (1,1).

Mathematics Subject Classifications: 53C41 · 53A10
Key words: Biharmonic curve · Harmonic curve · Involute curve · Rigid motions

INTRODUCTION

Firstly, Harmonic Maps Are Given as Follows: Let (M^m,g) and (N^n,h) be two Riemannian manifolds, the energy functional of a map $\phi \in C^1(M^m,N^n)$ is defined by

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2\, dv_g, \tag{1.1}$$

Where $|d\phi|$ is the Hilbert--Schmidt norm of the differential $d\phi$ and dv_g is the volume element on M. A map $\phi \in C^2(M^m,N^n)$ is called harmonic if it is a critical point of the energy functional, that is, if it is a solution of the Euler--Lagrange equationassociated to (1.1)

$$\tau(\phi) = \mathrm{Tr}_g \nabla d\phi = 0, \tag{1.2}$$

$\tau(\phi)$ is called the tension field of ϕ. Harmonic maps are solutions of a second order nonlinear elliptic system and they play a very important rôle in many branches of mathematics and physics where they may serve as a model for liquid crystal. We can refer to [1-3] for background on harmonic maps.

Secondly, Biharmonic Maps Are Given as Follows: A natural generalization of harmonic maps is given by integrating the square of the norm of the tension field. More precisely, the bi-energy functional of a map $\phi \in C^2(M^m,N^n)$ is defined by

$$E_2(\phi)=\frac{1}{2}\int_M |\tau(\phi)|^2\, dv_g, \tag{1.3}$$

a map $\phi \in C^4(M^m,N^n)$ is called biharmonic if it is a critical point of the bi-energy functional, that is, if it is a solution of the Euler--Lagrange equation associated to (1.3).

$$\tau_2(\phi) = -\mathrm{Tr}_g\left(\nabla^{\phi}\right)^2\tau(\phi) - \mathrm{Tr}_g R^N\left(\tau(\phi),d\phi\right)d\phi = 0, \tag{1.4}$$

Where

$$\mathrm{Tr}_g\left(\nabla^{\phi}\right)^2 = \sum_{i=1}^{m}\left(\nabla^{\phi}_{e_i}\nabla^{\phi}_{e_i} - \nabla^{\phi}_{\nabla^M_{e_i}e_i}\right) \tag{1.5}$$

for an orthonormal frame $\{e_1,e_2,..,e_m\}$ is the Laplacian on sections of the pull-back bundle ϕ^{-1} and R^N is the curvature operator on N.

There are a few results on biharmonic curves in arbitrary Riemannian manifolds. The biharmonic curves in the Heisenberg group Heis^3 are investigated in [4, 5] by Körpinar *et al.* and [6-10] Turhan *at al.*

In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions E(1,1). We characterize the timelike involute of spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions E(1,1).

Preliminaries: Let E(1,1) be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

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Topologically, E(1,1) is diffeomorphic to R^3 under the map

$$E(1,1) \to R^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \to (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \mathbf{X}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, [\mathbf{X}_2, \mathbf{X}_3] = 0, [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2.$$

Put

$$x^1 = x, x^2 = \frac{1}{2}(y+z), x^3 = \frac{1}{2}(y-z).$$

Then, we get

$$\mathbf{X}_1 = \frac{\partial}{\partial x^1}, \mathbf{X}_2 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} + e^{-x^1}\frac{\partial}{\partial x^3}\right),$$
$$\mathbf{X}_3 = \frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2} - e^{-x^1}\frac{\partial}{\partial x^3}\right). \quad (2.1)$$

The bracket relations are

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_3, [\mathbf{X}_2, \mathbf{X}_3] = 0, [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2. \quad (2.2)$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{X_1, X_2, X_3\}$ We consider left-invariant Lorentzian metric, given by

$$g = -\left(dx^1\right)^2 + \left(e^{-x^1}dx^2 + e^{x^1}dx^3\right)^2 + \left(e^{-x^1}dx^2 - e^{x^1}dx^3\right)^2, \quad (2.3)$$

Where

$$g(\mathbf{X}_1, \mathbf{X}_1) = -1, g(\mathbf{X}_2, \mathbf{X}_2) = g(\mathbf{X}_3, \mathbf{X}_3) = 1. \quad (2.4)$$

Let coframe of our frame be defined by

$$?^1 = dx^1, ?^2 = e^{-x^1}dx^2 + e^{x^1}dx^3, ?^3 = e^{-x^1}dx^2 - e^{x^1}dx^3.$$

Proposition 2.1.: For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{X}_3 & 0 & -\mathbf{X}_1 \\ \mathbf{X}_2 & -\mathbf{X}_1 & 0 \end{pmatrix}, \quad (2.5)$$

Where the (i,j)-element in the table above equals $\nabla_{X_i} X_j$ for our basis

$$\{\mathbf{X}_k, k = 1,2,3\} = \{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)W,Z).$$

Moreover we put

$$R_{ijk} = R(\mathbf{X}_i, \mathbf{X}_j)\mathbf{X}_k, R_{ijkl} = R(\mathbf{X}_i, \mathbf{X}_j, \mathbf{X}_k, \mathbf{X}_l),$$

Where the indices i,j,k and l take the values 1,2 and 3.

$$R_{121} = X_2, R_{131} = X_3, R_{232} = X_3$$

and

$$R_{1212} = -1, R_{1313} = -1, R_{2323} = -1. \quad (2.6)$$

Spacelike Biharmonic General Helices with Timelike Normal in the Lorentzian Group of Rigid Motions E(1,1): Let $\gamma: I \to E(1,1)$ be a non geodesics spacelike curve with timelike normal in the group of rigid motions E(1,1) parametrized by arc length. Let {T,N,B} be the Frenet frame fields on the group of rigid motions E(1,1). along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ) and B is chosen so that {T,N,B} is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_{T(s)}T(s) = \kappa(s)N(s),$$
$$\nabla_{T(s)}N(s) = \kappa(s)T(s) + \tau(s)B(s), \quad (3.1)$$
$$\nabla_{T(s)}B(s) = \tau(s)N(s),$$

Where $\kappa(s) = |\tau(\gamma)| = |\nabla_{T(s)}T(s)|$ is the curvature of γ, $\tau(s)$ is its torsion and

$$g(\mathbf{T}(s), \mathbf{T}(s)) = 1, g(\mathbf{N}(s), \mathbf{N}(s)) = -1, g(\mathbf{B}(s), \mathbf{B}(s)) = 1, \quad (3.2)$$
$$g(\mathbf{T}(s), \mathbf{N}(s)) = g(\mathbf{T}(s), \mathbf{B}(s)) = g(\mathbf{N}(s), \mathbf{B}(s)) = 0.$$

With respect to the orthonormal basis $\{X_1, X_2, X_3\}$ we can write

$$\mathbf{T}(s) = T_1(s)\mathbf{X}_1 + T_2(s)\mathbf{X}_2 + T_3(s)\mathbf{X}_3, \tag{3.3}$$

$$\mathbf{N}(s) = N_1(s)\mathbf{X}_1 + N_2(s)\mathbf{X}_2 + N_3(s)\mathbf{X}_3,$$

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = B_1(s)\mathbf{X}_1 + B_2(s)\mathbf{X}_2 + B_3(s)\mathbf{X}_3.$$

Theorem 3.1.: [11] $\gamma: I \to$E(1,1) is a non geodesic spacelike biharmonic curve with timelike normal in the Lorentzian group of rigid motions E(1,1) if and only if

$$\kappa(s) = \text{constant} \neq 0,$$

$$\kappa^2(s) + \tau^2(s) = 1 + 2B_1^2(s), \tag{3.4}$$

$$\tau'(s) = -2N_1(s)B_1(s).$$

Theorem 3.2.: [11] Let $\gamma: I \to$E(1,1) is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions E(1,1) Then, the parametric equations of γ are

$$x^1(\theta) = \sinh\wp\,\kappa s + a_1, \tag{3.5}$$

$$x^2(\theta) = \frac{\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\,\kappa s + a_1}}{2\left(\Lambda^2 + \sinh^2\wp\right)}\left[(\sinh\wp + \Lambda)\cos[\Lambda\kappa s + \Upsilon] + (\sinh\wp + \Lambda)\sin[\Lambda\kappa s + \Upsilon]\right] + a_2,$$

$$x^3(\theta) = \frac{\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\,\kappa s + a_1}}{2\left(\Lambda^2 + \sinh^2\wp\right)}\left[(\sinh\wp - \Lambda)\cos[\Lambda\kappa s + \Upsilon] + (\Lambda - \sinh\wp)\sin[\Lambda\kappa s + \Upsilon]\right] + a_3,$$

Where a_1, a_2, a_3 are constants of integration.

Timelike Involute Curve of Spacelike Biharmonic General Helices with Timelike Normal in the Lorentzian E(1,1)

Definition 4.1.: Let unit speed spacelike curve $\gamma: I \to$ E(1,1) and the curve $\beta: I \to$ E(1,1) be given. For $\forall_s \in I$, then the curve β is called the timelike involute of the curve γ, if the tangent at the point $\gamma(s)$ to the curve γ passes through the tangent at the point $\beta(s)$ to the curve β and

$$g(\mathrm{T}^*(s), \mathrm{T}(s)) = 0. \tag{4.1}$$

Let the Frenet-Serret frames of the curves γ and β be {T,N,B} and {T*,N*,B*}, respectively.

Theorem 4.2.: Let $\gamma: I \to$ E(1,1) be a unit speed spacelike biharmonic general helix with timelike normal and β its timelike involute curve on E(1,1). Then, the parametric equations of β are

$$x^1(s) = (\kappa s + \hat{o} - s)\sinh\wp + a_1,$$

$$x^2(s) = \frac{\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\,\kappa s + a_1}}{2\left(\Lambda^2 + \sinh^2\wp\right)}\left[(\sinh\wp + \Lambda)\cos[\Lambda\kappa s + \Upsilon] + (\sinh\wp + \Lambda)\sin[\Lambda\kappa s + \Upsilon]\right]$$

$$\frac{1}{2}\sqrt{1-\sinh^2\wp}\,(\hat{o} - s)e^{\sinh\wp\,\kappa s + a_1}(\cos[\Lambda\kappa s + \Upsilon] + \sin[\Lambda\kappa s + \Upsilon]) + a_2, \tag{4.2}$$

$$x^3(s) = \frac{\sqrt{1-\sinh^2\wp}\, e^{\sinh\wp\,\kappa s + a_1}}{2\left(\Lambda^2 + \sinh^2\wp\right)}\left[(\sinh\wp - \Lambda)\cos[\Lambda\kappa s + \Upsilon] + (\Lambda - \sinh\wp)\sin[\Lambda\kappa s + \Upsilon]\right]$$

$$\frac{1}{2}\sqrt{1-\sinh^2\wp}\,(\hat{o} - s)e^{-\sinh\wp\,\kappa s - a_1}(\cos[\Lambda\kappa s + \Upsilon] - \sin[\Lambda\kappa s + \Upsilon]) + a_3,$$

Where α_1, α_2, α_3, $\hat{o}$ are constants of integration.

Proof: Using Theorem 3.2, imply

$$\mathbf{T} = \sinh\wp\,\mathbf{e}_1 + \sqrt{1-\sinh^2\wp}\cos[\Lambda\kappa s + \Upsilon]\mathbf{e}_2 + \sqrt{1-\sinh^2\wp}\sin[\Lambda\kappa s + \Upsilon]\mathbf{e}_3. \tag{4.3}$$

Substituting (2.1) in (4.3), we obtain

$$\mathbf{T} = (\sinh\wp, \frac{1}{2}\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\,\kappa s + a_1}(\cos[\Lambda\kappa s + \Upsilon] + \sin[\Lambda\kappa s + \Upsilon]),$$

$$\frac{1}{2}\sqrt{1-\sinh^2\wp}\,e^{-\sinh\wp\,\kappa s - a_1}(\cos[\Lambda\kappa s + \Upsilon] - \sin[\Lambda\kappa s + \Upsilon])), \tag{4.4}$$

Where α_1 is constant of integration.

On the other hand, the curve $\beta(s)$ may be given as

$$\beta(s) = \gamma(s) + u(s)\mathrm{T}(s). \tag{4.5}$$

If we take the derivative (4.5), then we have

$$\beta'(s) = \left(1 + u'(s)\right)\mathbf{T}(s) + u(s)\kappa(s)\mathbf{N}(s).$$

Since the curve β is involute of the curve γ, $g(\mathrm{T}^*(s), \mathrm{T}(s)) = 0$. Then, we get

$$1 + u'(s) = 0 \text{or} u(s) = \hat{o} - s, \tag{4.6}$$

Where $\hat{o}$ is constant of integration.

Substituting (4.6) into (4.5), we get

$$\beta(s) = \gamma(s) + (\hat{o} - s)\mathbf{T}(s). \tag{4.7}$$

If we substitute (3.5) and (4.4) into (4.7), we have (4.2). Thus, the proof is completed.
We can use Mathematica for spacelike biharmonic general helix with timelike normal, yields

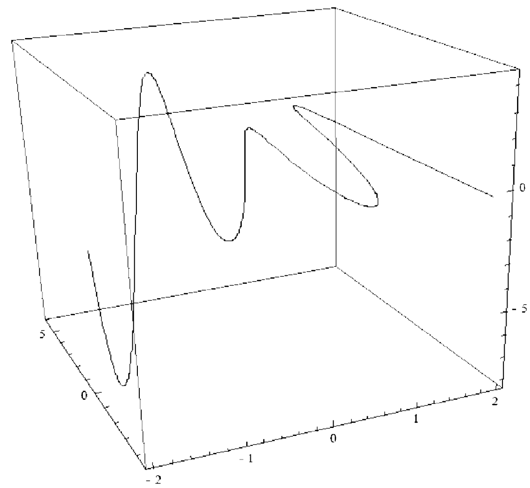

Fig. 1:

Similarly, we can use Mathematica for timelike involute curve, yields.

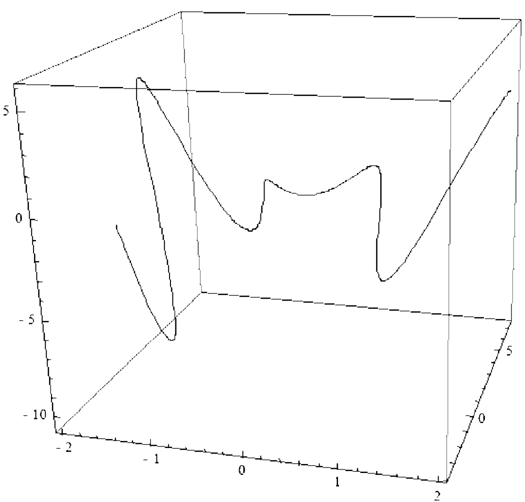

Fig. 2:

If we use Mathematica both spacelike biharmonic general helix with timelike normal and its timelike involute curve, we have.

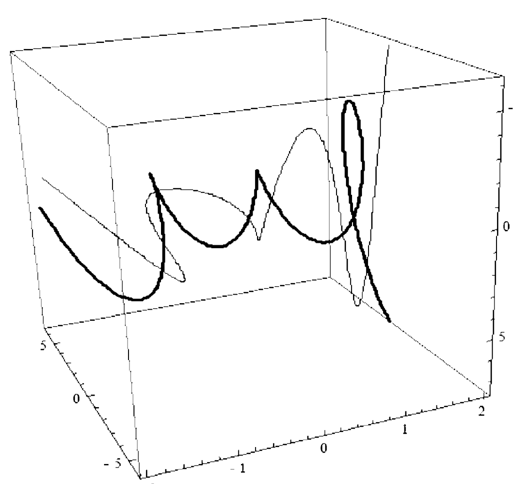

Fig. 3:

Lemma 4.3.: Let γ: $I \to$ E(1,1) is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions E(1,1) Then,

$$\kappa(s) = \sqrt{1+2B_1^2(s)}\cos\Sigma, \tag{4.8}$$

$$\tau(s) = \sqrt{1+2B_1^2(s)}\sin\Sigma,$$

Where Σ is arbitrary angle.

Proof: Using second equation of (3.4) we get (4.8).

Theorem 4.4.: Let γ: $I \to$ E(1,1) be a unit speed spacelike biharmonic general helix with timelike normal and β its timelike involute curve on E(1,1). Then, the parametric equations of β are

$$x^1(s) = \left(\sqrt{1+2B_1^2(s)}\cos\Sigma s + \hat{o} - s\right)\sinh\wp + a_1,$$

$$x^2(s) = \frac{\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\sqrt{1+2B_1^2(s)}\cos\Sigma s + a_1}}{2\left(\Lambda^2+\sinh^2\wp\right)}((\sinh\wp+\Lambda)\cos\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]$$

$$+(\sinh\wp+\Lambda)\sin\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right])$$

$$\frac{1}{2}\sqrt{1-\sinh^2\wp}(\hat{o}-s)e^{\sinh\wp\sqrt{1+2B_1^2(s)}\cos\Sigma s + a_1}(\cos\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]$$

$$+\sin\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]) + a_2, \tag{4.9}$$

$$x^3(s) = \frac{\sqrt{1-\sinh^2\wp}\,e^{\sinh\wp\sqrt{1+2B_1^2(s)}\cos\Sigma\kappa s + a_1}}{2\left(\Lambda^2+\sinh^2\wp\right)}((\sinh\wp-\Lambda)\cos\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]$$

$$+(\Lambda-\sinh\wp)\sin\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right])$$

$$\frac{1}{2}\sqrt{1-\sinh^2\wp}(\hat{o}-s)e^{-\sinh\wp\sqrt{1+2B_1^2(s)}\cos\Sigma s - a_1}(\cos\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]$$

$$-\sin\left[\Lambda\sqrt{1+2B_1^2(s)}\cos\Sigma s + \Upsilon\right]) + a_3,$$

Where α_1, α_2, α_3, $\hat{o}$ are constants of integration and Σ is arbitrary angle.

Proof: Substituting (4.8) into (4.2), we get (4.9). This completes the proof.

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