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Tower of Hanoi Problemwith Arbitrary Number of Pegs and Present a Solution

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Abstract: Suppose that we have m pegs and n disks of distinct sizes such that initially disks are stacked on the first peg ordred by size with the smallest at the top and the largest at the bottom. In this Paper we want to find a solution for transition disks to one of another pegs using the well-known movements of Tower of Hanoi such that there are no solution better than it yet.

Key words: Tower of Hanoi · Recursive formula · Pegs

INTRODUCTION

In the well-known Tower of Hanoi problem, composed over a hundred years ago by Lucas [1], a player is given 3 pegs and a certain number n of disks of distinct sizes and is required to transfer them from one peg to another. Initially all disks are stacked (composing a tower) on the first peg (the source) ordered by size, with the smallest at the top and the largest at the bottom. The goal is to transfer them to the third peg (the destination), moving only topmost disks and never placing a disk on top of a smaller one. The known recursive algorithm, which may be easily shown to be optimal, takes $h_n = 2^n - 1$ steps to accomplish the task. Since there is not muchmathematicalmystery left about the originalgame, its lovers developed various versions of it [2-6]. In this paper we want to solve Tower of Hanoi problem with arbitrary number of pegs.

Tower of Hanoi Problem with M Pegs: Suppose that we have n disks of distinct sizes $d_1,d...,d_n$, such that $d_1>,d_2 >... > d_n$ and m pegs (m>3). Initially disks are stacked on the first peg ordred by size (Fig.1.). The goal is to transfer all disks to one of another pegs.

Algorithm of Solution: For transfer of disks from peg 1 to one of another pegs sush as peg m, at the first d_1 must be placed in the bottom of the peg m. it is possible when all other disks be stacked in another m-2 pegs. then d_2 must be transfered to peg m, top of d_1. But it is possible when d_2 be in one peg alone, for example peg 2 and all disks except d_1 and d_2 be stacked in another m-3 pegs. In this manner for transition of d_{m-2} to peg m top of $d_1,d_2,...,d_{m-3}$ disks must have been a position as following. (Fig.2.)

That means $n-(m-2)$ remainder disks must be stacked on one peg such as peg $m-1$ in Fig.2. certainly. In fact we want to find a recursive formula using this idea for solving this problem.

If we show the number of motions for transition of n disks to the peg m with H_n^m, it is obvious that if $n < m$ then $H_n^m = 2n-1$.

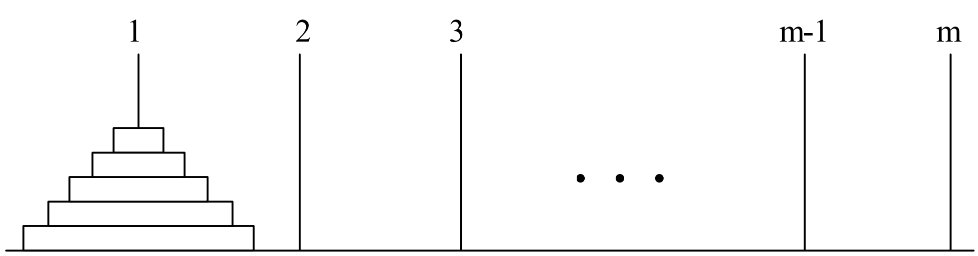

Fig. 1: The board of the Tower of Hanoi problem with m

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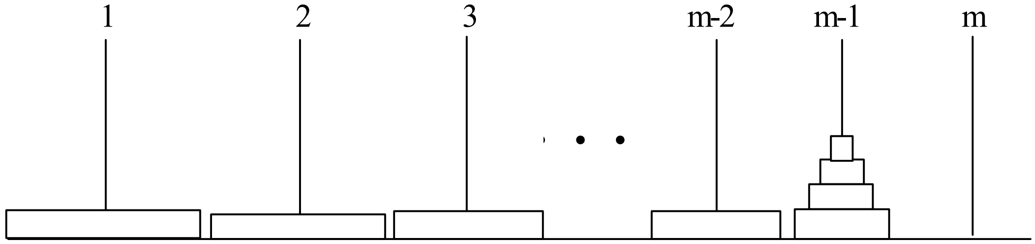

Fig. 2: Steps of transition of disks

Now with this introductions we want to find H_n^m for $n \geq m$, using following steps:

Step 1: Transition of disks d_{m-3},..., d_n, with $H_{n-(m-2)}^m$ motions from peg 1 to peg m–1.

Step 2: Transition of disks $d_2,d_3,...,d_{m-2}$, from peg 1 to pegs 2,3,..., m–2 with m–3 motions

Step 3: Transition of disks $d_1,d_2,d_3,...,d_{m-2}$, to peg m with m–2 motions.

Step 4: Transition of disks d_{m-3},..., d_n, with $H_{n-(m-2)}^m$ motions from peg m–1 to peg m.

Therefore for ≥,we have:

$$H_n^m = H_{n-(m-2)}^m + (m-3) + (m-2) + H_{n-(m-2)}^m$$
$$\Rightarrow H_n^m = 2H_{n-(m-2)}^m + 2m - 5$$

Upper relation is a non-homogeneous recursive formula of degree m–2 and we want to solve it with primal conditions:

$$H_0^m = 0,\ H_1^m = 1,\ H_2^m = 3\ ,\ \ldots\ ,H_{m-3}^m = 2(m-3)-1$$

We must solve this recursive formula in m–2 cases:

Case 1: $n = (m-2)k$
Case 2: $n = (m-2)k + 1$
Case 3: $n = (m-2)k + 2$
⋮ ⋮
Case m-2: $n = (m-2)k + (m-3)$

In all upper cases is a non-negative integer number.

Studing of Case 1: In this case $n = (m$–$2)$ and we have:

$$\begin{aligned} H_n^m &= 2H_{n-(m-2)}^m + 2m - 5 \\ &= 2(2H_{n-2(m-2)}^m + 2m-5) + 2m - 5 = 2^2 H_{n-2(m-2)}^m + 3(2m-5) \\ &\quad \vdots \qquad\qquad\qquad \vdots \\ &= 2^k H_{n-k(m-2)}^m + (2^k - 1)(2m-5) = 2^k H_0^m + (2^k-1)(2m-5) \end{aligned}$$
$$\Rightarrow H_n^m = (2^{\frac{n}{m-2}} - 1)(2m-5)$$

Studing of Case 2: In this case $n = (m–2)k + 1$ and we have:

$$H_n^m = 2^k H_1^m + (2^k - 1)(2m-5) = 2^k + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = 2^{\frac{n-1}{m-2}} + (2^{\frac{n-1}{m-2}} - 1)(2m-5)$$

Studing of Case 3: In this case $n = (m–2)k + 2$ and we have

$$H_n^m = 2^k H_2^m + (2^k - 1)(2m-5) = (2^k \times 3) + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = 3 \times 2^{\frac{n-2}{m-2}} + (2^{\frac{n-2}{m-2}} - 1)(2m-5)$$

Studing of Case m-2: In this case $n = (m–2)k + (m–3)$ and we have:

$$H_n^m = 2^k H_{m-3}^m + (2^k - 1)(2m-5) = 2^k (2(m-3)-1) + (2^k - 1)(2m-5)$$
$$\Rightarrow H_n^m = (2(m-3)-1)2^{\frac{n-(m-3)}{m-2}} + (2^{\frac{n-(m-3)}{m-2}} - 1)(2m-5)$$

Therefore in the abstract we have:

If $n = (m–2)k$ then $H_n^m = (2^{\frac{n}{m-2}} - 1)(2m-5)$

If $n = (m–2)k + 1$ then $H_n^m = 2^{\frac{n-1}{m-2}} + (2^{\frac{n-1}{m-2}} - 1)(2m-5)$

If $n = (m–2)k + 2$ then $H_n^m = 3 \times 2^{\frac{n-2}{m-2}} + (2^{\frac{n-2}{m-2}} - 1)(2m-5)$

$\vdots$

If $n = (m–2)k + (m–3)$ then $H_n^m = (2(m-3)-1)2^{\frac{n-(m-3)}{m-2}} + (2^{\frac{n-(m-3)}{m-2}} - 1)(2m-5)$

Concluding Remarks: We found a solution for transfer of n disks useing the well-known movements of Tower of Hanoi with m pegs from one peg to one of another pegs. We did not find optimal solution, but there are no solution better than this solution yet.

In general case, H_n^m can be obtained as following:

$$H_n^m = \begin{cases} (2m-5)(2^{\frac{n}{m-2}} - 1) & i = 0 \\ (2i-1)\ 2^{\frac{n-i}{m-2}} + (2m-5)(2^{\frac{n-i}{m-2}} - 1) & i \neq 0 \end{cases}$$

Such that i is remainder of division of by m–2 ($i = n$ mod(m–)).

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