

Run the GMRES with Ayachour

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Abstract: GMRES method is a good and strong method for solving large Linear systems of equations, but its convergence speed is low in some problems. This GMRES. The only difference is that it doesn't use Gunz vocations to compute least – square in stead of it, this minimal value is computed by using of continuous multi variables functions. The example given shows the efficiency of this implementation.

Key words: GMRES method • Ayachour

INTRODUCTION

One of the important computational problems in the applied science and engineering is the solution of linear system of equations $Ax = b$.

For solving a nonsymmetric linear system $Ax = b$, several methods have been developed see [2-4].

One of the most popular method is the GMRES method or “The generalized minimal residual method”.

This method by using the Arnoldi process makes an orthonormal basis $V = [v_1, v_2, \dots, v_k]$ in Krylov subspace. $K_k(A, r_0) = \text{span}\{r_0, Ar_0, \dots, A^{k-1}r_0\}$ where $r_0 = b - Ax_0$ and x_0 is an initial suggestion. See [5].

Ayachour Implementation: Ayachour proposed this method in 2003[1]. This method also uses Gram smith orthogonal- making process (Algorithm 2-2) to produce a single orthonormal basis for $k_k(A,)$. It is implemented as following:

Assume that $H_k \in \mathbb{C}^{k \times k}$ is upper- triangular matrix.

$$H_k = \begin{pmatrix} h_{2,1} & h_{2,2} & \dots & h_{2,k} \\ & h_{3,2} & \dots & h_{3,k} \\ & & \ddots & \vdots \\ & & & h_{k+1,k} \end{pmatrix}$$

And V_k is a matrix of $n \times k$ which its columns are $v_i \in i = 1, \dots, k$ and $\bar{H}_k = \begin{pmatrix} w \\ H_k \end{pmatrix}$ matrix with $w = (h_{1,1} \dots h_{1,k})$.

Here we are seeking for $y_k \in \mathbb{C}^k$:

$$\|r_0 - AV_k y_k\| = \min_{y_k \in \mathbb{C}^k} \|r_0 - AV_k y_k\| = \min_{y_k \in \mathbb{C}^k} \|e_1 - \bar{H}_k y_k\|_2$$

$e_1 = (1, 0, \dots, 0)^T \in \mathbb{C}^{(k+1) \times 1}$

To do this, we should first define the internal multi placion of vectors as following:

$$\forall x, y \in \mathbb{R}^n : \langle x, y \rangle = x^* y = \sum_{i=1}^n \bar{x}^{(i)} y^{(i)}$$

$\bar{x}^{(i)}, x^*$ means vector x component i th pair vector x pair transposed, respectively).

Now, by wising above mentioned multi placion , we have from [1]:

$$\|e_1 - \bar{H}_k y\|_2^2 = 1 - \langle w^*, y \rangle - \langle y, w^* \rangle + \langle w^*, y \rangle \langle y, w^* \rangle + \langle H_k y, H_k y \rangle \quad (1)$$

We should consider two following modes to minimize the value of equation (1):

1: Assume that $h_{k+1,k} \neq 0$ so H_k is an even & upper-triangular matrix. so that by placing $u = H_k^{*-1} w^*$, $t = H_k y$, the equation (1) changes

$$\|e_1 - \bar{H}_k y\|_2^2 = 1 - \langle u, t \rangle - \langle t, u \rangle + \langle u, t \rangle \langle t, u \rangle + \langle t, t \rangle$$

If we assume that function of $f_k: \mathbb{C}^k \rightarrow \mathbb{R}$ Has been defined as

$$f_k(t) = 1 - \langle u, t \rangle - \langle t, u \rangle + \langle u, t \rangle \langle t, u \rangle + \langle t, t \rangle$$

We should find minimal value of f_k to minimize (6-3). f_k is the differentiable function whose differential is indicate by df_k . after simplifying these venations, we have:

$$df_k(h) = \text{Re}(\langle (t, u) - 1, u + t, h \rangle)$$

$R(z)$ means the true part of z mixed number. If t' is the absolute minimum of f_k , then

$$(\langle t', u \rangle - 1)u + t' = 0 \quad (2)$$

And from the equation (2) is shown that t', u are in parallel. If we assume that $t' = cu$, that according to (2), $c = \frac{1}{(1 + \|u\|^2)}$ is obtained.

Finally,

$$\frac{\|r_k\|}{\|r_0\|} = \min_{y \in K_k} \|e_1 - \bar{H}_k y\|_2,$$

is minimized by $y_k = H_k^{-1} t'$ & an approximate residual norm

is obtained by $\frac{\|r_k\|}{\|r_0\|} = \sqrt{f_k(t')} = \sqrt{c}$ [1].

2: Assume that k , $h_{k+1,k} = 0$ is the smallest script that occurs in this mode. It is showed in such a way that k is the degree of v_1 minimal multi expression. Now, you assume that H_k^t has been defined as $H_k + e_k e_k^T$, in other

words, H_k^t is the same as H_k that we have placed number 1 in stead of $h_{k+1,k}$, so H_k^t is even. By placing $u = H_k^{*-1} w^*$, $t = H_k^t y$ the equation (1) changes to

$$\|e_1 - \bar{H}_k y\|_2^2 = 1 - \langle u, t \rangle - \langle t, u \rangle + \langle u, t \rangle \langle t, u \rangle + \langle t, t \rangle - \langle t, e_k \rangle \langle e_k, t \rangle$$

as before, if assume that $g_k: \mathbb{C}^k \rightarrow \mathbb{R}$ is as following

$$g_k(t) = 1 - \langle u, t \rangle - \langle t, u \rangle + \langle u, t \rangle \langle t, u \rangle + \langle t, t \rangle - t^{(k)} \bar{t}^{(k)}$$

Where $t^{(k)}$ is k th component of vector t . as before, in this situation, we should find absolute minimum $g_k(t)$ by using its derivative (dg_{k_i}) . so we have from [1].

$$dg_{k_i}(h) = \text{Re} \left(\left((\langle t, u \rangle - 1)u + t - t^{(k)} e_k, h \right) \right)$$

If t' is g_k absolute minimum, then it is true in the following equation: $((\langle t', u \rangle - 1)u + t' - t'^{(k)} e_k)_{e_k} = 0$ resulted from it: $((\langle t', u \rangle - 1)u^{(k)}) = 0$. Now, by returning from $j = 1$ to $j = k$ we have coefficients β_1, \dots, β_k from \mathbb{C} as if

$$v'_{j+1} = \sum_{i=1}^j \beta_j A^i v_1 - \bar{u}^{(i)} v_1 \quad (4)$$

$$v'_{j+1} = h_{j+1,j} v_{j+1}, \beta_1 = h_{2,1}^{-1} h_{3,2}^{-1} \dots h_{j,j-1}^{-1} \neq 0$$

$$h_{j+1,j} = \|v'_{j+1}\| \neq 0$$

k has been considered as the degree of minimal multi expressions $h_{k+1,k} = 0$, $v_1, u^{(k)}$ can't be zero so according to [1]. It is showed that $\langle t', u \rangle = 1$ by using $\langle t', u \rangle$. (4), it is indicated that $t'_k = \frac{1}{u_k}$, $t'_i = 0$, $i = 1, \dots, k-1$. finally, the equation (1) is minimized by $y_k = H_k^{-1} t'$. in this situation, we obtain the accurate solution $Ax = b$.

Agachour in [1] has proposed the theorem introduces the algorithm which is an integration of two a above mentioned mode for obtaining the solution of equation and doesn't study an algorithm separately for each mode.

The following algorithm is based on [1], where R_k has been used instead of H_k^{r-1} And $R_k(:, k)$ is k th column from the matrix R_k and finally we have:

1. Choose x then $r = b - Ax$ and $\rho_0 = \|r\|$. if $\rho_0 \leq \epsilon ps$ then x is a solving of (3-1),

Stop, else set $v_1 = \frac{r}{\|r\|}$, $\alpha_0 = 1$

2. for $k = 1, \dots, m$ do

a. Evaluate $v_{k+1} = Av_k$

b. Set $w_k = \langle v_{k+1}, v_1 \rangle$, $v_{k+1} = v_{k+1} - w_k v_1$

c. for $j = 1, \dots, k$ do

$$\lambda_{j-1} = \langle v_{k+1}, v_j \rangle, v_{k+1} = v_{k+1} - \lambda_{j-1} v_j$$

d. $\beta = \|v_{k+1}\|$, $v_{k+1} = v_{k+1}/\beta$, set $g = (\lambda_1, \dots, \lambda_{k-1})^T$.

e. Evaluate $R_k = \begin{pmatrix} R_{k-1} & -R_{k-1}g \\ & 1 \end{pmatrix}$.

$$u_k = \langle R_k(:, k), w^T \rangle$$

f. Set $\gamma_k = \frac{1}{\sqrt{\beta^2 + (u_k \alpha_0)^2}}$, $\sin \theta_k = \beta \gamma_k$, $\alpha_1 = \alpha_0 \sin \theta_k$

g. $\|r_k\| = \rho_0 \alpha_1$, if $\|r_k\| < \epsilon ps$ or $|u_k| < \epsilon ps$ go to 3.

h. Update $u_k = \frac{u_k}{\beta}$, $R_k(:, k) = \frac{1}{\beta} R_k(:, k)$, $\alpha_0 = \alpha_1$

Let k be the final iteration number from 2.

- Set $y = \left(\sin^2 \theta_k u_1, \dots, \sin^2 \theta_k u_{k-1}, \gamma_k^2 u_k \right)^T$,
 $z^{(k)} = H_k^{-1} y$
- $x = x + \rho_0 \alpha_0^2 V_k z^{(k)}$, $r = b - Ax$
- If $\|r\| < \text{eps}$ accept x otherwise $\rho_0 = \|r\|$, $v_1 = \frac{r}{\|r\|}$ and return to 1.

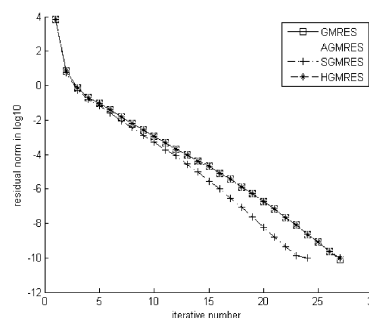
Numerical example: In this part, a well known nominal example has been proposed for GMRES implementations. In the curve figure, residual error r has been drawn as $\log_{10} r$ based on corresponding iteration number. If $\|r\| < 10^{-10}$ then methods stop and it has been considered that $x_0 = (0, 0, \dots, 0)^T$.

Example 1: In this example, matrix A has been chosen from [6] and vector b is considered as if $x = (1, 1, \dots, 1)^T \in \mathbb{R}^n$ is a solution of the Linear system of equation $Ax = b$ and the from of matrix A is as following:

$$A = \begin{pmatrix} 1 & 0.21 & 1.2 & 0 & 0.13 & 1.45 & 0 & 0 & \dots & 0 \\ 0.45 & 2 & 0.21 & 1.2 & 0 & 0.13 & 1.42 & \ddots & & 0 \\ 0 & 0.45 & 3 & 0.21 & 1.2 & 0 & 0.13 & \ddots & \ddots & \vdots \\ 0.12 & 0 & 0.45 & 4 & 0.21 & 1.2 & 0 & \ddots & \ddots & 0 \\ 0.11 & 0.12 & 0 & 0.45 & 5 & 0.21 & 1.2 & \ddots & \ddots & 1.42 \\ 0 & 0.11 & 0.12 & 0 & 0.45 & 6 & 0.21 & \ddots & \ddots & 0.13 \\ 0 & 0 & 0.11 & 0.12 & 0 & 0.45 & 7 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1.2 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & n-1 & 0.21 \\ 0 & 0 & \dots & 0 & 0 & 0.11 & 0.12 & 0 & 0.45 & n \end{pmatrix}$$

The cure related to residual norm in \log_{10} scale is mentioned for this example that shows the behavior of each implementations of the previous part.

In this example $n = 500$, & $k = 15$ for GMRES (k). As it is showed from the figure 1, two methods of AGMRES & GMRES have represented a similar behavior but SCMRES method becomes convergent faster. The interesting point in these examples is that we will solve a problem with a dimension about 15 whose solving is more convenient, in less than 30 iterations, rather than to solve a problem with the dimension 500 and we will abstain the solution of initial problem through it. In this example, SGMRES becomes convergent faster (furring 24 iterations) and other 3 implementations represent a similar behavior as if the residual norm in \log_{10} is integrated for them.



	Gmres	Agmres	Sgmres	Hgmres
Iteration	27	27	24	27
Error	0.0551×10^{-01}	9.7374×10^{-01}	9.3613×10^{-01}	9.7373×10^{-01}

Fig. 1: The residual error figure for the example 1. The following table shows nominal results of these implementations until converging on expected solution with an accuracy less than 10^{-10} :

CONCLUSION

This paper has studied Ayachaur in this way. This method is similar to that of standard GMRES. The only difference is that it doesn't use Gunz vocations to compute least-square $\|e_1 - \bar{H}_k y\|_2$ in stead of it, this minimal value is computed by using of continuous multi variables functions. The example given shows the efficiency of this implementation.

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