

## Two New Preconditioners for Solving Linear Systems

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**Abstract:** In the recent years a number of preconditioners have been applied to the linear systems, e.g., [1], [2]. In this paper, we present two new preconditioners  $(I + F_\alpha)$  and  $(I + H_\beta)$ . We also provide some sufficient conditions for convergence of the preconditioned Gauss-Seidel method for the Z-matrices. Numerical examples show that the spectral radii of the new methods are better comparing with those preconditioners  $(I + S_\alpha)$  and  $(I + K_\beta)$ .

**Key words:** Preconditioning Gauss-Seidel method • Spectral radii • Z and M-matrices

### INTRODUCTION AND PRELIMINARIES

Let us consider the iterative methods for the solution of the linear system

$$Ax=b, x, b \in R^{n \times n} \quad (1)$$

Where  $A = [\alpha_{ij}] \in R^{n \times n}$ . The basic iterative method for solving the linear system (1) is

$$Mx_{k+1} = Mx_k + b \quad k = 0, 1, \dots \quad (2)$$

Where  $x_0$  is an initial vector,  $A=M-N$  and  $M$  is nonsingular. Then (2) can also be written as

$$Mx_{k+1} = M^{-1}Nx_k + M^{-1}b, \quad k = 0, 1, \dots$$

Where  $M^{-1}N$  is called an iteration matrix of the iterative method. It is well known that the iterative method converges to the exact solution of (1) for any initial vector  $x_0$  if and only if  $\rho(M^{-1}N) < 1$ , where  $\rho(M^{-1}N)$  denotes the spectral radius of  $M^{-1}N$ .

We now transform the original linear system (1) into the preconditioned linear system

$$PAX = Pb \quad (3)$$

Where  $P$  is called a preconditioner. Then the basic iterative method for solving the linear system (3) is

$PM = M_p - N_p$  and  $M_p$  is nonsingular. Clearly  $M_p^{-1}N_p$  is an iteration matrix of the iterative method (3).

In the recent years, many efforts have been done to provide preconditioners, for example in 1997, H. Niki suggested a preconditioner as  $P_\alpha = (I + S_\alpha)$  [1], Also in 2008, Q. Liu represented preconditioner as  $P_\beta = (I + K_\beta)$  for H-matrices [2].

In this paper, we represent two preconditioners (a) and (b). In (a) we have two diagonals above main diagonal and in (b) two diagonals below the main diagonal. In section 3, we investigate sufficient conditions for convergence of the preconditioned Gauss-Seidel method when coefficient matrix be the Z-matrices. In the end by numerical examples, we show that the spectral radius derived by our preconditioned matrix works better.

**Proposed Methods (a) and (b):** without loss of generality, let the matrix of the the linear system (1) be  $A = I - L - U$ , where  $I$  is an identity matrix,  $L$  and  $U$  strictly lower and Upper triangular matrices obtained from  $A$ , respectively.  
 (a) If  $\alpha_{ii+1} \neq 0$  &  $\alpha_{ii+2} \neq 0$ , then consider the preconditioned linear system

$$A_\alpha x = b_\alpha$$

Where

$$P_\alpha = (I + F_\alpha), A_\alpha = (I + F_\alpha)A \text{ and } b_\alpha = (I + F_\alpha)b$$

With

$$F_{\alpha'} = \begin{bmatrix} 0 & -\alpha'_1 a_{12} & -\alpha'_1 \alpha'_{13} & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\alpha'_2 \alpha'_{23} & -\alpha'_2 a_{24} & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & -\alpha'_{n-2} a_{n-1n} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

And  $\alpha'_i \geq 0 \quad i=1, 2, \dots, n-1$

$$\bar{\alpha}_{ij} = \begin{cases} a_{ij} - \alpha'_i a_{ii+1} \alpha'_{i+1j} - \alpha'_i a_{ii+2} a_{i+2j} & 1 \leq i < n, 1 \leq j \leq n \\ 1 - \alpha'_i a_{ii+1i} - \alpha'_i a_{ii+2} a_{i+2i} & i = j \\ a_{ij} & i = n \end{cases}$$

$A_{\alpha'}$  can be written as follows:

$$A_{\alpha'} = [\bar{a}_{ij}] = \bar{D}_{\alpha'} - \bar{L}_{\alpha'} - \bar{U}_{\alpha'} = (I - L - \bar{D} - \bar{L}) - (U - F_{\alpha'} + F_{\alpha'} U) = M_{\alpha'} - N_{\alpha'}$$

Where  $\bar{D}, \bar{L}$  are the diagonal and strictly lower triangular parts of  $F_{\alpha'} L$  respectively. Also when

$$\alpha'_i a_{ii+1} a_{i+1i} + \alpha'_i a_{ii+2} a_{i+2i} \neq 1, \quad (i=1, 2, \dots, n-1)$$

Then  $(M_{\alpha'})^{-1}$  exists and  $(M_{\alpha'})^{-1} \geq 0$  for  $\alpha' \leq 1$ . So it is Possible to define the Gauss-Seidel iteration matrix for  $A_{\alpha'}$  as:

$$T_{\alpha'} = (I - L - \bar{D} - \bar{L})^{-1} (U - F_{\alpha'} + F_{\alpha'} U)$$

**Remark 2.1:** The splitting  $A_{\alpha'} = M_{\alpha'} - N_{\alpha'} (\alpha' > 1)$  is not the regular splitting,because  $N_{\alpha'} (\alpha' > 1)$  is not the nonnegative matrix. If assume that  $\alpha'_{n-1} \alpha'_{n-1n} = \alpha'_{n-1n}$  and also two following conditions

$$\begin{cases} a_{ii+1} a_{i+1i+2} > |a_{ii+2}| \\ a_{ii+2} a_{i+2i+1} > |a_{ii+1}| \end{cases}$$

are available, then we conclude that  $N_{\alpha'} > 0$  ( $\alpha' > 1$ ) and also  $N_{\alpha'}^{-1} N_{\alpha'} \geq 0$  ■

(b) If  $\alpha'_{ii-1} \neq 0 \& \alpha'_{ii-2} \neq 0, 2 \leq i \leq n$  then consider the preconditioned linear system

$$A_{\beta'} \mathbf{x} = b_{\beta'}$$

Where  $P_{\beta'} = (I + H_{\beta'})$ ,  $A_{\beta'} = (I + H_{\beta'}) A$  and  $b_{\beta'} = (I + H_{\beta'}) b$ ,

$$\text{with } \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -\beta'_2 a_{21} & 0 & \cdots & \cdots & 0 \\ -\beta'_3 a_{31} & -\beta'_3 a_{32} & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & -\beta'_n a_{nn-2} & -\beta'_n a_{nn-1} & 0 \end{bmatrix} \text{ and } \beta'_i \geq 0 \quad i=2, 3, \dots, n$$

$$\tilde{a}_{ij} = \begin{cases} a_{ij} & i=1 \\ a_{ij} - \beta'_i a_{ii-1} a_{i-1j} - \beta'_i a_{ii-2} a_{i-2j} & 1 < i \leq n \quad 1 \leq j \leq n \\ 1 - \beta'_i a_{ii-1} a_{i-1j} - \beta'_i a_{ii-2} a_{i-2j} & i = j \end{cases}$$

We can write  $A_{\beta'}$  as follows:

$$A_{\beta'} = [\tilde{a}_{ij}] = \tilde{D}_{\beta'} - \tilde{L}_{\beta'} - \tilde{U}_{\beta'} = (I - L - \tilde{D} + H_{\beta'}) - (U + \tilde{U}) = M_{\beta'} - N_{\beta'}$$

Where  $\tilde{D}, \tilde{L}, \tilde{U}$  are the diagonal and strictly upper and strictly lower triangular parts of  $H_{\beta'} U$  respectively and  $(M_{\beta'})^{-1}$  exists, also  $(M_{\beta'})^{-1} \geq 0$  for  $(0 \leq \beta' \leq)$  and for  $\beta' > 1$  we see that  $(M_{\beta'})^{-1} \geq 0$ . It is Possible to define the Gauss-Seidel iteration matrix for  $A_{\beta'}$  as:

$$T_{\beta'} = (I - L - \tilde{D} - \tilde{L} + H_{\beta'} - H_{\beta'} L)^{-1}(U + \tilde{U})$$

**Convergence of the Proposed Method for Nonsingular Diagonally Dominant Z-matrices:** In this section we prove the convergence of our methods.

**Lemma 3.1[1]:** An upper bound for the spectral radius of the Gauss-Seidel iteration matrix  $T$  is given by:

$$\rho(T) \leq \max_i \frac{\bar{u}_i}{1 - l_i}$$

Where  $l_i, u_i$  are sums of elements in the  $i$ th row of the triangular matrices  $L, U$ , respectively.

**Theorem 3.2:** Let  $A$  be a nonsingular diagonally dominant Z-matrix with unit diagonal elements. Then if

$$\sum_{j=1}^n a_{ij} = 0, (i \neq n) \text{ we have } \sum_{j=i+1}^n a_{i+1j} + a_{i+2j} > 0 \text{ and } \rho(T_{\alpha}) < 1 \quad (0 \leq \alpha' \leq).$$

**Proof:** If  $A$  is a diagonally dominant Z-matrix, then for  $1 \leq i \leq n$ , we have

$$\begin{aligned} 0 \leq \alpha_{ii+1} \alpha_{i+1j} &\leq 1 \quad j \neq i+1 \\ 0 \leq \alpha_{ii+2} \alpha_{i+2j} &\leq 1 \quad j \neq i+2 \end{aligned}$$

Let

$$\begin{aligned} p_i &= \alpha_{ii+1} \alpha_{i+1j} \quad j \neq i+1 \\ s_i &= \alpha_{ii+2} \alpha_{i+2j} \quad j \neq i+2 \end{aligned}$$

$$\begin{aligned} v_i &= a_{ii+1} \sum_{j=1}^{i-1} a_{i+1j} + a_{ii+2} \sum_{j=1}^{i-1} a_{i+2j} \\ w_i &= a_{ii+1} \sum_{j=i+1}^n a_{i+1j} + a_{ii+2} \sum_{j=i+1}^n a_{i+2j} \end{aligned}$$

therefore  $p_i + s_i + v_i + w_i < 0$ .

Now assume  $(\bar{d}_{\alpha'})_i, (\bar{l}_{\alpha'})_i, (\bar{u}_{\alpha'})_i$  are sums of the elements in row i of  $\bar{D}_{\alpha'}, \bar{L}_{\alpha'}, \bar{U}_{\alpha'}$  and  $l_i, u_i$  the sums of the elements in row ith, ( $i=1, 2, \dots, n-1$ ) of L, U respectively.

$$\begin{aligned} (\bar{d}_{\alpha'})_i &= 1 - \alpha'_i p_i - \alpha'_i s_i \\ (\bar{l}_{\alpha'})_i &= \sum_{j=1}^{i-1} |\bar{a}_{ij}| = L_i + \alpha'_i v_i \\ (\bar{u}_{\alpha'})_i &= \sum_{j=i+1}^n |\bar{a}_{ij}| = u_i + \alpha'_i w_i \end{aligned}$$

If  $0 \leq \alpha'_i \leq 1$  and

$$\begin{aligned} 1 - \alpha'_i a_{ii+1} a_{i+1j} - \alpha'_i a_{ii+2} a_{i+2j} &> 0 \quad \text{for } i=j \\ a_{ij} - \alpha'_i a_{ii+1} a_{i+1j} - \alpha'_i a_{ii+2} a_{i+2j} &\leq 0 \quad \text{for } i \neq j \end{aligned}$$

Then,  $(\bar{l}_{\alpha'})_i, (\bar{u}_{\alpha'})_i \geq 0$  and  $A_{\alpha'}$  is a Z-matrix, moreover we have

Hence  $A_{\alpha'}$  is strictly diagonally dominant.

To prove convergence, by  $(d_{\alpha'})_i - (l_{\alpha'})_i \geq 0$  and lemma 3.1 we have  $\rho(T_{\alpha'}) \leq 1$ . ■

**Theorem 3.3:** Let A be a matrix defined as in Theorem 3.2, now if

$$\alpha'' = \min_i \frac{1 - l_i - u_i - 2a_{ii+1} - 2a_{ii+2}}{p_i + s_i + v_i + w_i - 2a_{ii+1} - 2a_{ii+2} - 2a_{ii+1}a_{i+1i+2} - 2a_{ii+2}a_{i+2i+1}}, \quad 1 \leq i \leq n.$$

Then  $\alpha'' > 1$  and for  $0 \leq \alpha' < \alpha''$ ,  $\rho(T_{\alpha'}) > 1$ .

**Proof:** Since  $1 - l_i - u_i > 0$  ( $1 \leq i \leq n$ ), then

$$1 - l_i - u_i - 2a_{ii+1} - 2a_{ii+2} > p_i + s_i + v_i + w_i - 2a_{ii+1} - 2a_{ii+2} - 2a_{ii+1}a_{i+1i+2} - 2a_{ii+2}a_{i+2i+1} > 0$$

Hence  $\alpha'' > 1$ . Now Let

$$(\bar{\bar{u}}_{\alpha'})_i = \sum_{j=i+1}^n |a_{ij} - \alpha'_i a_{ii+1} a_{i+1j} - \alpha'_i a_{ii+2} a_{i+2j}|$$

Therefore

$$\begin{aligned} (\bar{\bar{u}}_{\alpha'})_i &= \sum_{j=i+1}^n |(1 - \alpha') a_{ii+1} + (1 - \alpha') a_{ii+1} a_{i+1j} - \alpha' a_{ii+1} a_{i+1j} - \alpha' a_{ii+2} a_{i+2j}| \\ &\leq |(1 - \alpha') a_{ii+1} - \alpha' a_{ii+1} a_{i+1i+2}| + |(1 - \alpha') a_{ii+2} - \alpha' a_{ii+2} a_{i+2i+1}| + \\ &\quad \sum_{j=i+3}^n |a_{ij} - \alpha' a_{ii+1} a_{i+1j} - \alpha' a_{ii+2} a_{i+2j}| = 2(1 - \alpha') a_{ii+1} + 2(1 - \alpha') a_{ii+2} + \\ &\quad 2\alpha' a_{ii+1} a_{i+1i+2} + 2\alpha' a_{ii+2} a_{i+2i+1} - \sum_{j=i+1}^n (a_{ij} - \alpha' a_{ii+1} a_{i+1j} - \alpha' a_{ii+2} a_{i+2j}) = \\ &\quad (u_i + 2a_{ii+1} + 2a_{ii+2}) + \alpha'(w_i - 2a_{ii+1} - 2a_{ii+2}) + 2\alpha' a_{ii+1} a_{i+1i+2} + 2\alpha' a_{ii+2} a_{i+2i+1}. \end{aligned}$$

Then

$$\begin{aligned} (d_{\alpha'})_i - (l_{\alpha'})_i - (\bar{\bar{u}}_{\alpha'})_i &= (1 - \alpha' p_i - \alpha' s_i) - (l_i + \alpha' v_i) - (u_i + 2a_{ii+1} + 2a_{ii+2}) - \\ &\quad \alpha'(w_i - 2a_{ii+1} - 2a_{ii+2} - 2a_{ii+1}a_{i+1i+2} - 2a_{ii+2}a_{i+2i+1}) > (1 - l_i - u_i - 2a_{ii+1} - 2a_{ii+2}) - \\ &\quad \alpha''(p_i + s_i + v_i + w_i - 2a_{ii+1} - 2a_{ii+2} - 2a_{ii+1}a_{i+1i+2} - 2a_{ii+2}a_{i+2i+1}) = 0 \end{aligned}$$

Therefore  $A_{\alpha'}$  is a strictly diagonally dominant matrix and with Lemma 3.1  $\rho(T_{\alpha'}) < 1$  ■

**Theorem 3.4:** Let A be a nonsingular diagonally dominant Z-matrix with unit diagonal elements. If

$$\sum_{j=1}^n a_{ij} = 0, i \neq n \text{ then } \sum_{j=1}^{i-1} a_{i-2,j} + a_{i-2,j} > 0 \text{ and } \rho(T_{\beta}) < 1 (0 \leq \beta' \leq 1).$$

**Proof:** If A is a diagonally dominant Z-matrix, then for  $1 < i \leq n$ , have

$$\begin{aligned} -1 &\leq \alpha_{ii-2} \alpha_{i-2,j} < 0 & j \neq i-2 \\ -1 &\leq \alpha_{ii-1} \alpha_{i-1,j} < 0 & j \neq i-1 \end{aligned}$$

Let

$$\begin{aligned} m_i &= \alpha_{ii-2} \alpha_{i-2,j} & j \neq i-2 \\ q_i &= \alpha_{ii-1} \alpha_{i-1,j} & j \neq i-1 \end{aligned}$$

$$\begin{aligned} n_i &= a_{ii-2} \sum_{j=1}^{i-1} a_{i-2,j} + a_{ii-1} \sum_{j=1}^{i-1} a_{i-1,j} \\ r_i &= a_{ii-2} \sum_{j=i+1}^n a_{i-2,j} + a_{ii-1} \sum_{j=i+1}^n a_{i-1,j} \end{aligned}$$

$$\text{Therefore } m_i + q_i + n_i + r_i = a_{ii-2} \sum_{j=1}^n a_{i-2,j} + a_{ii-1} \sum_{j=1}^n a_{i-1,j} < 0$$

Now let  $(\tilde{d}_{\beta})_i, (\tilde{l}_{\beta})_i, (\tilde{u}_{\beta})_i$ , ( $1 < i \leq n$ ), be the sums of the elements in ith row of  $(\tilde{D}_{\beta}), (\tilde{L}_{\beta}), (\tilde{U}_{\beta})$ , respectively.

$$\begin{aligned} (\tilde{d}_{\beta})_i &= 1 - \beta'_i m_i - \beta'_i q_i \\ (\tilde{l}_{\beta})_i &= 1_i + \beta'_i n_i \\ (\tilde{u}_{\beta})_i &= u_i + \beta'_i r_i \end{aligned}$$

$$\text{If } 0 \leq \beta'_i \leq 1, \text{ then } 1 - \beta'_i a_{ii-2} a_{i-2,j} - \beta'_i a_{ii-1} a_{i-1,j} > 0, \quad i = j$$

And if  $a_{ij} - \beta'_i a_{ii-2} a_{i-2,j} - \beta'_i a_{ii-1} a_{i-1,j} \leq 0, \quad i \neq j$  then  $(\tilde{l}_{\beta'})_i, (\tilde{u}_{\beta'})_i \geq 0$ ,  $A_{\beta'}$  is a Z-matrix, moreover have

$$(d_{\beta'})_i - (l_{\beta'})_i - (u_{\beta'})_i = (1 - l_i - u_i) - \beta'_i (m_i + q_i + n_i + r_i) > 0, \quad 1 < i \leq n$$

Hence,  $(A_{\beta'})$  strictly dominant diagonally. For prove convergence, have:

$(d_{\beta})_i - (l_{\beta})_i > (u_{\beta})_i \geq 0$  and by lemma 3.1  $\rho(T_{\beta}) < 1$  ■

**Theorem 3.5:** Let A be a matrix defined as in Theorem 3.4, if

$$\beta'' = \min_i \frac{1 - l_i - u_i - 2a_{ii-2} - 2a_{ii-1}}{m_i + q_i + n_i + r_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2,i-1} - 2a_{ii-1}a_{i-1,i-2}}, \quad 1 < i \leq n$$

then  $\beta'' > 1$  and for  $0 \leq \beta' < \beta''$ ,  $\rho(T_{\beta'}) < 1$ .

**Proof:** We first prove  $\beta'' > 1$ , consider

$$m_i + q_i + n_i + p_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2i-1} - 2a_{ii-1}a_{i-1i-2} =$$

$$\begin{aligned} a_{ii-2} \sum_{j=1}^n a_{i-2j} - 2a_{ii-2} - 2a_{ii-2}a_{i-2i-1} + a_{ii-1} \sum_{j=1}^n a_{i-1j} - 2a_{ii-1} - 2a_{ii-1}a_{i-1i-2} = \\ a_{ii-2} \left( \sum_{\substack{j=1 \\ j \neq i-2}}^n a_{i-2j} - 1 - a_{i-2i-1} \right) + a_{ii-1} \left( \sum_{\substack{j=1 \\ j \neq i-1}}^n a_{i-1j} - 1 - a_{i-1i-2} \right) > 0 , 1 < i \leq n . \end{aligned}$$

Since  $1 - l_i - u_i > 0$  and

$$1 - l_i - u_i - 2a_{ii-2} - 2a_{ii-1} > m_i + q_i + n_i + p_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2i-1} - 2a_{ii-1}a_{i-1i-2} > 0$$

$$\text{Hence } \beta'' > 1. \text{ Let } (\tilde{\tilde{I}}_{\beta'})_i = \sum_{j=1}^{i-1} |a_{ij} - \beta' a_{ii-2} a_{i-2j} - \beta' a_{ii-1} a_{i-1j}|$$

$$\begin{aligned} (\tilde{\tilde{I}}_{\beta'})_i = & |(1 - \beta') a_{ii-2} - \beta' a_{ii-1} a_{i-1i-2} - (1 - \beta') a_{ii-1} - \beta' a_{ii-2} a_{i-1i-2}| + \\ & \sum_{j=1}^{i-3} |a_{ij} - \beta' a_{ii-2} a_{i-2j} - \beta' a_{ii-1} a_{i-1j}| \leq 2(1 - \beta') a_{ii-2} + 2(1 - \beta') a_{ii-1} + 2\beta' a_{ii-2} a_{i-1i-1} + \\ & 2\beta' a_{ii-1} a_{i-1i-2} - \sum_{j=1}^{i-1} (a_{ij} - \beta' a_{ii-2} a_{i-2j} - \beta' a_{ii-1} a_{i-1j}) = (l_i + 2a_{ii-2} + 2a_{ii-1}) + \\ & \beta'(n_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2i-1} - 2a_{ii-1}a_{i-1i-2}) \end{aligned}$$

Thus

$$\begin{aligned} (d_{\beta'})_i - (\tilde{\tilde{I}}_{\beta'})_i - (u_{\beta'})_i = & (1 - \beta'm_i - \beta'q_i) - (l_i + 2a_{ii-2} + 2a_{ii-1}) - \\ & \beta'(n_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2i-1} - 2a_{ii-1}a_{i-1i-2}) - \\ & (u_i + \beta'p_i) > (1 - l_i - u_i - 2a_{ii-2} - 2a_{ii-1}) - \\ & \beta''(m_i + q_i + n_i + p_i - 2a_{ii-2} - 2a_{ii-1} - 2a_{ii-2}a_{i-2i-1} - 2a_{ii-1}a_{i-1i-2}) = 0 \end{aligned}$$

Therefore  $A_{\beta'}$  is a strictly diagonally dominant matrix and from Lemma 3.1 have  $\rho(T_{\beta'}) < 1$  ■

**Remark 3.1:** For estimate  $\beta'$  in  $P_{\beta} = I + K_{\beta}$  we work similar to  $P_{\beta'} = (I + H_{\beta'})$ , but  $K_{\beta'}$  represent one diagonal above main diagonal. Also in  $\tilde{a}_{ij}$  and theorems 3.4, 3.5 put  $a_{ii-2} = 0$ .

**Theorem 3.6.[3, 4]:** Let Abe a nonnegative matrix.then:

- (a) If  $\gamma_1 x \leq Ax$  for Some nonnegative vector x,  $x \neq 0$ , then  $\gamma_1 \leq \rho(A)$
- (b) If  $\leq \gamma_2$  for Some vector x, then  $\rho(A) \leq \gamma_2$ . Moreover, if A is irreducible and if  $\gamma_1 x \leq Ax \leq \gamma_2 x$  For Some nonnegative vector x, then  $\gamma_1 \leq \rho(A) \leq \gamma_2$  and x is a positive vector.

**Theorem3.7.[5]:** Let  $A=M_1-N_1=M_2-N_2$  be two regular splitting of A, where  $A^{-1} \geq 0$ . If  $M_1^{-1} \geq M_2^{-1}$ , then  $1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \geq 0$

Particular if  $M_1^{-1} > M_2^{-1}$  and  $A^{-1} > 0$  then  $1 > \rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 0$

**Theorem 3.8.[5]:** Let  $A=M_1-N_1=M_2-N_2$  be two regular splitting of  $A$ , where  $A^{-1} \geq 0$ , then  $N_2 \geq N_1$  implies that  $M_1^{-1} \geq M_2^{-1}$ .

**Theorem 3.9.[6]:** Let  $A \geq 0$  be an  $n \times n$  matrix. Then the following hold:

- (a)  $A$  has a nonnegative real eigenvalue equal to its spectral  $\rho(A)$ . Moreover, this eigenvalue is positive unless  $A$  is reducible and the normal from of  $A$  is strictly upper triangular.
- (b)  $A$  has an eigenvector  $x \geq 0$  corresponding to  $\rho(A)$ .
- (c)  $\rho(A)$  does not decrease when any entry of  $A$  is increased.

### Comparison Theorems for Z-matrices and M-matrices

**Theorem 4.1:** If  $A=[a_{ij}]$  be a Z-matrix and  $T, T_\alpha$  are iteration matrices of the Gauss-Seidel method and preconditioner Gauss-Seidel method, respectively. When  $0 \leq \alpha' \leq 1$  and by remark 2.1 for  $0 \leq \alpha' < \alpha''$  we have

- a)  $\rho(T_\alpha) < 1$  if  $\rho(T) < 1$
- b)  $\rho(T_\alpha) = 1$  if  $\rho(T) = 1$
- c)  $\rho(T_\alpha) > 1$  if  $\rho(T) > 1$

**Proof:** Since  $T = (I-L)^{-1} U \geq 0$  therefore by Theorem 3.9 there exists a vector  $x$  such that  $x \geq 0$  as  $Tx = \rho(T)x$ .

Since  $T_\alpha$ ,  $0 \leq \alpha' \leq 1$  is nonnegative matrix and also from remark 2.1  $T_\alpha$  ( $0 \leq \alpha' < \alpha''$ ) is nonnegative matrix, then exist vector  $x$  such that  $x \geq 0$  and that  $T_\alpha x = \rho(T_\alpha)x$ .

Now

$$\begin{aligned} T_{\alpha'} &= (I - L - \bar{D} - \bar{L})^{-1} (U - F_{\alpha'} + F_{\alpha'} U) \\ &= (I - L - \bar{D} - \bar{L})^{-1} (I + F_{\alpha'}) U x - (I - L - \bar{D} - \bar{L})^{-1} (F_{\alpha'}) x \\ &= (I - L - \bar{D} - \bar{L})^{-1} (I + F_{\alpha'}) \rho(T) (I - L) x - (I - L - \bar{D} - \bar{L})^{-1} (F_{\alpha'}) x \end{aligned}$$

Therefore

$$\begin{aligned} T_{\alpha'} x - Tx &= (I - L - \bar{D} - \bar{L})^{-1} [\rho(T) (I + F_{\alpha'}) (I - L) x - (F_{\alpha'}) x - \\ &\quad (I - L - \bar{D} - \bar{L}) (I - L)^{-1} U x] = (I - L - \bar{D} - \bar{L})^{-1} [\rho(T) (I - L) x + (\rho(T) - 1) F_{\alpha'} x - U x] = \\ &= (I - L - \bar{D} - \bar{L})^{-1} (\rho(T) - 1) (F_{\alpha'}) x \end{aligned}$$

Since  $(M)_\alpha^{-1}$  and  $(F_\alpha)$  are nonnegative matrices. Then by theorem 2.2 [3] we have

If  $\rho(T) < 1$  then  $\rho(T_\alpha) < 1$

If  $\rho(T) = 1$  then  $\rho(T_\alpha) = 1$

If  $\rho(T) > 1$  Then  $\rho(T_\alpha) > 1$  ■

**Theorem 4.2:** If  $A = [\tilde{a}_{ij}]$  be a Z-matrix and  $T, T_\beta$  are iteration matrices of the Gauss-Seidel method and preconditioner Gauss-Seidel method, respectively. For  $0 \leq \beta' \leq 1$  we have

- a)  $\rho(T_\beta) < 1$  if  $\rho(T) < 1$
- b)  $\rho(T_\beta) = 1$  if  $\rho(T) > 1$
- c)  $\rho(T_\beta) > 1$  if  $\rho(T) > 1$

**Proof:** We have  $Uy = \rho(T)(I-L)y$ , then

$$[(L + U - I) + (1 - \rho(T))I + (\rho(T) - 1)L]y = 0. \text{ Therefore } (L + U - I)y = (\rho(T) - 1)(I - L)y$$

Since  $T_\beta$  is nonnegative matrix and also  $\rho(T_\beta) > 1$ , therefore  $T_\beta y = \rho(T_\beta)y$ .

$$\begin{aligned}
 T_{\beta'} y - T y &= (\tilde{D} - \tilde{L})^{-1} \left[ -\rho(T)(I - \tilde{D}) + \rho(T)(L + H_{\beta'} L - H_{\beta'}) + (U + \tilde{U}) \right] y \\
 &= (\tilde{D} - \tilde{L})^{-1} \left[ (\rho(T) - 1)(\tilde{D}) + (\rho(T) - 1)(H_{\beta'} L - H_{\beta'}) + (\tilde{D} + H_{\beta'} L - H_{\beta'} + \tilde{U}) \right] y \\
 &= (\tilde{D} - \tilde{L})^{-1} \left[ (\rho(T) - 1)(\tilde{D}) + (\rho(T) - 1)(H_{\beta'} L - H_{\beta'}) + (\rho(T) - 1)H_{\beta'}(I - L) \right] y \\
 &= (\tilde{D} - \tilde{L})^{-1} \left[ (\rho(T) - 1)(\tilde{D}) \right] y
 \end{aligned}$$

Now, If  $\rho(T) < 1$  then  $\rho(T_{\beta'}) < 1$

If  $\rho(T) = 1$  then  $\rho(T_{\beta'}) = 1$

If  $\rho(T) > 1$  then  $\rho(T_{\beta'}) < 1$  ■

**Lemma 4.3:** Let A be an irreducibly diagonally dominant Z-matrix, where  $A = M - N$  and  $A_{\alpha'} = M_{\alpha'} - N_{\alpha'}$  are Gauss-Seidel splitting. Then we have the following inequalities  $M_{\alpha'>1}^{-1} \geq M_{\alpha'=1}^{-1} \geq M_{\alpha'<1}^{-1} \geq M^{-1} \geq 0$

**Proof:** Since  $(I - \bar{D})^{-1} \geq 0$ , for  $1 \geq \alpha' < \alpha''$  and  $(L + \bar{L}) \geq 0$  then we have the following inequalities

$$M_{\alpha'}^{-1} = \left[ I + (I - \bar{D})^{-1}(L + \bar{L}) + \left\{ (I - \bar{D})^{-1}(L + \bar{L}) \right\}^2 + \dots + \left\{ (I - \bar{D})(L + \bar{L}) \right\}^{n-1} \right] (I - \bar{D}) \geq 0$$

Now if  $0 \leq \alpha' \leq 1$  then  $(I - \bar{D}) \geq I$  and  $(L + \bar{L}) \geq L$ ,

thus  $M_{\alpha'=1}^{-1} \geq M_{\alpha'<1}^{-1} \geq M^{-1} \geq 0$ .

If  $\alpha' > 1$  since  $(I - \bar{D}) \geq (I - \bar{D}_{(\alpha'=1)})^{-1} \geq 0$  and  $(L + \bar{L}_{\alpha'>1}) \geq (L + \bar{L}_{\alpha'=1}) \geq 0$ ,

thus  $M_{\alpha'>1}^{-1} \geq M_{\alpha'=1}^{-1} \geq 0$ . ■

**Theorem 4.4:** Let A be an irreducibly diagonally dominant Z-matrix, where  $A = M - N$  and  $A_{\alpha'} = (I + F_{\alpha'}) = M_{\alpha'} - N_{\alpha'} (0 \leq \alpha' \leq 1)$  are Gauss-Seidel regular splitting. Then

$$\rho(M_{\alpha'=1}^{-1}N_{\alpha'=1}) \leq \rho(M_{\alpha'<1}^{-1}N_{\alpha'<1}) \leq \rho(M_{\alpha'}^{-1}N_{\alpha'}) < 1$$

Also, with remark 2.1,  $A_{\alpha'} = M_{\alpha'} - N_{\alpha'} (0 \leq \alpha' < \alpha'')$  is Gauss-Seidel regular splitting. Then we have

$$\rho(M_{\alpha'>1}^{-1}N_{\alpha'>1}) \leq \rho(M_{\alpha'=1}^{-1}N_{\alpha'=1}) \leq \rho(M_{\alpha'<1}^{-1}N_{\alpha'<1}) \leq \rho(M_{\alpha'}^{-1}N_{\alpha'}) < 1.$$

**Proof:** Since  $M_{\alpha'}^{-1} \geq 0, N_{\alpha'} \geq 0, (0 \leq \alpha' \leq 1)$  and from remark 2.1 we have  $M_{\alpha'}^{-1} \geq 0, N_{\alpha'} \geq 0, (0 \leq \alpha' < \alpha'')$ , therefore from Lemma 4.3 and theorem 3.7, the proof is completed. ■

**Lemma 4.5:** Let A be an irreducibly diagonally dominant Z-matrix, where  $A = M - N$  and  $A_{\beta'} = I + H_{\beta'} = M_{\beta'} - N_{\beta'} (0 \leq \beta' \leq 1)$  are Gauss-Seidel regular splitting. Then the following inequalities hold,  $M_{\beta'=1}^{-1} \geq M_{\beta'<1}^{-1} \geq M^{-1} \geq 0$ .

**Proof:** Since  $0 \leq N_{\beta>1} \leq N_{\beta=1} \leq N_{\beta<1} \leq N$  and from Theorem 3.9 too, the proof is completed. ■

**Theorem 4.6:** Let A be an irreducibly diagonally dominant Z-matrix, where  $A = M - N$  and  $A_{\alpha'} = A_{\beta'} - A_{\beta}, (0 \leq \beta' \leq )$  are Gauss-Seidel regular splitting. Then  $\rho(M_{\beta'=1}^{-1}N_{\beta'=1}) \leq \rho(M_{\beta'<1}^{-1}N_{\beta'<1}) \leq \rho(M_{\beta'}^{-1}N_{\beta'}) < 1$

**Proof:** Since  $N_{\beta'} \geq 0$ , then the inequalities are hold. ■

**Theorem 4.7:** Let  $A$  be an irreducibly diagonally dominant Z-matrix. Let  $A_{\alpha=1} = M_{\alpha=1} - N_{\alpha=1} = (I + S_{\alpha=1})A$ ,  $A_{\alpha'=1} = M_{\alpha'=1} - N_{\alpha'=1} = (I + F_{\alpha'=1})A$ , are Gauss-Seidel regular then  $\rho(M_{\alpha=1}^{-1}N_{\alpha'=1}) \leq \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1$

**Proof:** From theorem 4.5.  $\rho(M_{\alpha'=1}^{-1}N_{\alpha'=1}) < 1$

Consider any fixed vector  $e > 0$  (e.g. with all component equal to 1) and  $v = A^{-1}e$  (no row of  $A^{-1}$  can have all null entries), then  $v > 0$ .

We have the following equation:

$$(A_{\alpha=1} - A_{\alpha=1})v = (F_{\alpha=1} - S_{\alpha=1})Av = (F_{\alpha=1} \bullet S_{\alpha=1})e \geq 0, \text{ therefore } A_{\alpha=1}v \geq A_{\alpha=1}v$$

Then the following relation holds:

$$M_{\alpha'>1}^{-1}N_{\alpha'=1}v = (I - M_{\alpha>1}^{-1}N_{\alpha>1})v \geq M_{\alpha=1}^{-1}N_{\alpha=1}v = (I - M_{\alpha=1}^{-1}N_{\alpha=1})v$$

Which theorem 3.6 implies  $\rho(M_{\alpha'=1}^{-1}N_{\alpha'=1}) \leq \rho(M_{\alpha=1}^{-1}N_{\alpha=1})$ . From [4] we have  $\rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1$ .

Therefore  $\rho(M_{\alpha'=1}^{-1}N_{\alpha'=1}) \leq \rho(M_{\alpha=1}^{-1}N_{\alpha=1}) < 1$ .

Also by [4] we have

$$M_{\alpha}^{-1} = \left[ L + \left\{ (I - D_1)^{-1}(L + E) \right\} + \left\{ (I - D)^{-1}(L + E) \right\}^2 + \dots + \left\{ (I - D_{\alpha})^{-1}(L + E) \right\}^{n-1} \right] (I - D_{\alpha=1})^{-1} \geq 0$$

Therefore  $M_{\alpha'>1}^{-1} \geq M_{\alpha=1}^{-1}$  and by theorem 3.7 the proof is completed. ■

**Theorem 4.8:** Let  $A$  be an irreducibly diagonally dominant Z-matrix. Let  $A_{\beta} = M_{\beta=1} - N_{\beta=1} = (I + K_{\beta=1})A$ ,  $A_{\beta'} = M_{\beta'=1} - N_{\beta'=1} = (I + H_{\beta'=1})A$ , are Gauss-Seidel regular splitting there exists a positive vector  $\sigma$  such that  $0 \leq A_{\beta=1}\sigma \leq N_{\beta'}\sigma$ . Then

$$\rho(M_{\beta'=1}^{-1}N_{\beta'=1}) \leq \rho(M_{\beta=1}^{-1}N_{\beta=1}) < 1$$

**Remark 4.1:** From Definition 1.3 [4] and corollary 1.10[4] an irreducibly diagonally dominant Z-matrix is an M-matrix.

**Remark 4.2:** It is difficult to estimate  $\alpha'_{opt}$ ,  $\beta'_{opt}$  by theorem 3.3 and 3.5 and only they obtain by numerical computations.

**Remark 4.3:** It is difficult to compare the spectral radii of the two different iteration matrices  $(M_{\alpha'>1}^{-1}N_{\alpha'>1}), (M_{\alpha>1}^{-1}N_{\alpha>1})$  and  $(M_{\beta'>1}^{-1}N_{\beta'>1}), (M_{\beta>1}^{-1}N_{\beta>1})$  which are not nonnegative [except conditions said in remark 2.1]. Also we cannot compare  $\alpha$  with  $\alpha'$  and  $\beta$  with  $\beta'$ , therefore their spectral radii are not comparable.

**Numerical Examples:** In the following tables we consider.

EST : Estimate cases, according to said relations.

$A_1$  : General matrix.

$A_2$  : Strictly upper triangular matrices elements norm more than strictly lower triangular matrices elements norm.

$A_3$  : Strictly lower triangular matrices elements norm more than strictly upper triangular matrices elements norm.

$A_4$  : Symmetric matrix.

#### Comparison of Spectral Radii:

$$A_1 = \begin{cases} 1 & i = j \\ -1/(i+1/3*j+n) & i \neq j \end{cases}$$

Preconditioners

n	I	I+S	I+F	I+K	I+H	$\alpha_{opt}$	$\alpha'_{opt}$	$\beta_{opt}$	$\beta'_{opt}$
5	0.2310	0.1315	0.0849	0.2180	0.1977	1.7	1.27	4.85	3.21
10	0.3159	0.2540	0.2058	0.3159	0.3053	2.7	1.93	22	13.60
20	0.3655	0.3325	0.3027	0.3655	0.3655	5.38	3.66	41.9	45.40
40	0.3923	0.3754	0.3594	0.3923	0.3923	10.83	5.78	77	63.00

$I + S_\alpha$	$I + F\alpha'$	$I + K_\beta$	$I + H_{\beta'}$	$\alpha_{EST}$	$\alpha'_{EST}$	$\beta_{EST}$	$I + S_{EST}$	$I + F_{\alpha EST}$	$I + K_{\beta EST}$
0.0786	0.0300	0.1870	0.1588	3.29	2.84	3.64	0.1687	0.2887	0.1942
0.1100	0.0688	0.2538	0.2531	3.37	2.57	3.74	0.1243	0.1326	0.3020
0.1381	0.1225	0.2954	0.3037	3.94	2.71	6.35	0.2191	0.1686	0.3596
0.1527	0.1463	0.2984	0.3748	5.33	3.33	5.73	0.2949	0.2726	0.3907

$$A_2 = \begin{cases} 1 & i = j \\ -1/(i+10*j) & i > j \\ -1/(j+100*i) & i < j \end{cases}$$

Preconditioners

n	I	I+S	I+F	$\alpha_{opt}$	$I + S_\alpha$	$\alpha'_{opt}$	$I + F_{\alpha'}$
7	0.0110	0.0067	0.0042	2.00	0.0031	1.32	0.0019
10	0.0157	0.0114	0.0085	2.42	0.0042	1.67	0.0025
20	0.0281	0.0238	0.0207	5.10	0.0051	2.88	0.0041
40	0.0450	0.0410	0.0380	8.49	0.0107	4.91	0.0084

$I + K$	$\beta_{opt}$	$I + K_\beta$	$\beta'_{opt}$	$I + H_{\beta'}$
0.0107	8.00	0.0094	5.00	0.0081
0.0155	11.00	0.0141	7.00	0.0130
0.0279	22.00	0.0260	12.00	0.0248
0.0448	33.50	0.0409	18.00	0.0414

$$A_3 = \begin{cases} 1 & i = j \\ -1/(i+100*j) & i > j \\ -1/(j+10*i) & i < j \end{cases}$$

Preconditioners

n	I	I+S	I+F	$\alpha_{opt}$	$I + S_\alpha$	$\alpha'_{opt}$	$I + F_{\alpha'}$
7	0.0339	0.0148	0.0018	1.31	0.0099	1.09	0.0047
10	0.0490	0.0283	0.0168	1.56	0.0151	1.26	0.0080
20	0.0839	0.0632	0.0494	2.74	0.0248	1.71	0.0212
40	0.1257	0.1064	0.0929	4.64	0.0395	2.70	0.0352

$I + K$	$I + H$	$\beta_{opt}$	$I + K_\beta$	$\beta'_{opt}$	$I + F_{\alpha'}$
0.0337	0.0330	26.60	0.0291	11.00	0.0262
0.0488	0.0483	40.00	0.0435	22.70	0.0389
0.0837	0.0834	65.50	0.0774	52.10	0.0714
0.1255	0.1253	107.00	0.1184	88.50	0.1120

$$A_4 = \begin{cases} 1 & i = j \\ -1/(i+j+n) & i \neq j \end{cases}$$

Preconditioners

n	I	I+S	I+F	I+K	I+H	$\alpha_{opt}$	$I + S_\alpha$	$\alpha'_{opt}$	$I + F_{\alpha'}$	$I + K_{\beta_{opt}}$
5	0.1654	0.0850	0.0504	0.1557	0.1403	1.54	0.0521	1.22	0.0190	4.86
10	0.1780	0.2332	0.1371	0.2300	0.2245	2.41	0.0759	1.73	0.0473	20.00
20	0.2740	0.2428	0.2155	0.2730	0.2713	4.66	0.0973	3.38	0.0925	49.00
40	0.2964	0.2799	0.2645	0.2961	0.2956	9.24	0.1087	4.98	0.1041	90.00

$I + K_\beta$	$\beta'_{opt}$	$I + H_{\beta'}$	$\alpha_{EST}$	$I + S\alpha_{EST}$	$\alpha'_{EST}$	$I + F_{\alpha'EST}$	$\beta_{EST}$	$I + K_{\beta_{EST}}$
0.1319	3.09	0.1112	4.70	0.2622	3.73	0.3712	4.76	0.1323
0.1999	13.50	0.1803	5.24	0.1732	3.70	0.1846	5.37	0.2193
0.2243	51.00	0.2273	4.35	0.1445	6.85	0.1347	7.02	0.2683
0.2438	73.50	0.2791	6.02	0.1182	10.36	0.1203	10.57	0.2940

We compare spectral radii with value arbitrary less than one (0.9) in first example

$$A_1 = \begin{cases} 1 & i = j \\ -1/(i+1/3*j+n) & i \neq j \end{cases}$$

#### Preconditioners

n	$I + S_{\alpha=0.9}$	$I + F_{\alpha=0.9}$	$I + K_{\beta=0.9}$	$I + H_{\beta=0.9}$
5	0.1422	0.1002	0.2192	0.2006
10	0.2605	0.2178	0.2707	0.3063
20	0.3359	0.3093	0.3645	0.3627
40	0.3771	0.3627	0.3920	0.3915

#### CONCLUSION

- The spectral radius of two preconditioners  $I + S_{\alpha=EST}$  and  $I + K_{\beta=EST}$  are not comparable in general in the sense that, sometimes one of them works better than the other and sometime not.
- If  $\alpha_{EST} < 1$  then the spectral radius is less than one, but consider that it may be possible the spectral radius of the preconditioned matrix be greater than of the original one.
- In the symmetric matrices the spectral radius of preconditioner  $I + S_{\alpha}$  is better than  $I + S_{\alpha=EST}$  and also of  $I + F_{\alpha}$  is better than  $I + F_{\alpha=EST}$ .
- If the absolute values of entries located in blow and the above of the main diagonal are almost equal, then again we have the properties mentioned in 3.
- The preconditioner  $I + F_{\alpha}$  and  $I + H_{\beta}$  are better than the preconditioner of  $I + S_{\alpha}$  and  $I + K_{\beta}$  respectively.
- When  $\alpha, \alpha', \beta, \beta'$  (arbitrary 0.9), the spectral radius becomes less than one.

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