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Six Order Iterative Method for Solving Nonlinear Equations

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Abstract: In this paper, a new six order iterative method is derived for solving nonlinear equations, which is an improvement of a method prescribed by Cordero and Torregrosa [5]. Several numerical examples are given to illustrate the efficiency and performance of the new method.

AMS subject Classification codes: 65H05 · 65B09
Key words: Newton-Raphson Method · Order of convergence · Iterative method

INTRODUCTION

Solving non-linear equations is one of the most important problems in numerical analysis. In recent years, several iterative methods have been proposed and analyzed for finding the numerical solutions of nonlinear equation $f(x)$, [1-3] and the reference therein. In this paper, we consider three step iterative methods to find a simple root α, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a non-linear equation $f(x_n)$, where $f : R \to R$, be continuously differentiable real valued function.

Newton's method for a single non-linear equation is defined by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

This is an important and basic method [4], which converges quadratically.

MATERIALS AND METHODS

In this section, we describe our proposed method for finding a simple root α of $f(x)$ by improving a method prescribed by Cordero and Torregrosa [5]. The following definition is needed for convergence of our method.

Definition: (Order of Convergence) [6]: Let $\alpha \in R, x_n \in R$, n=0, 1, 2…. Then the sequence $\{x_n\}$ is said to converge to α if

$$\lim_{n\to\infty} |x_n - \alpha| = 0$$

In addition, there exists a constants $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n > n_0$

$$|x_{n+1} - \alpha| \leq |x_n - \alpha|^p$$

then $\{x_n\}$ is said to convergence to α of order p.

The Open Newton's method [5], which has third-order convergence, is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

$$x_{n+1} = y_n - \frac{3f(x_n)}{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})}, \quad \text{n=0, 1, 2....} \tag{3}$$

Starting with an initial approximation x_0 in the neighborhood of the root α this method converges cubically to α.

Using the idea proposed by [6], we construct a three step iterative method which is a modification of the method given by the equation (2) and (3) as follows:

For n=0, 1, 2 … define

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$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{4}$$

$$z_n = x_n - \frac{3f(x_n)}{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})} \tag{5}$$

and

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \tag{6}$$

Now, using the linear approximation on two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$ we get

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n)$$

Thus an approximation to $f'(z_n)$ is given by

$$f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n} f'(y_n) + \frac{z_n - y_n}{x_n - y_n} f'(x_n)$$

Now using (4) and (5), we get

$$f'(z_n) \approx \frac{f'(x_n)\{3f'(y_n) - 3f'(x_n) + 2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})\}}{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})} \tag{7}$$

Now replacing $f'(z_n)$ in (6) by (7) we get the following algorithm:

Algorithm RM: INPUT initial approximation x_0; tolerance ε; maximum number of iterations N_0.
OUTPUT approximate solution x_{n+1}, or message of failure.

Step 1 : Set n=0 and i=1.
Step 2 : While $i \le N_0$ do steps 3-5.
Step 3 : Calculate

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{8a}$$

$$z_n = y_n - \frac{3f(x_n)}{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})} \tag{8b}$$

$$x_{n+1} = z_n - \frac{f(z_n)\left\{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})\right\}}{f'(x_n)\left\{3f'(y_n) - 3f'(x_n) + 2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})\right\}} \tag{8c}$$

Step 4 : If $|x_n - x_{n+1}| < \varepsilon$; then OUTPUT ($x_{n+1}$); stop.
Step 5 : Set n=n+1; i=i+1 and go to Step 2.
Step 6 : OUTPUT ('Method failed after N_0 iterations, N_0='N_0); stop.

Convergence Analysis: In this section, we shall establish the six-order convergence of proposed method in this paper (i.e Algorithm RM).

Theorem 3.1: Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the iterative method defined by Algorithm RM has sixth-order convergence.

Proof: Let α be a simple zero of f. Since f is sufficiently differentiable, by expanding $f(x)$ and $f'(x)$ about α, we get

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)]$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 4c_5 e_n^4 + O(e_n^5)] \tag{9}$$

Where $c_j = \frac{f^{(j)}(\alpha)}{j!\, f'(\alpha)}$ for j=2, 3… and $e_i = x_i - \alpha,\ \ i = n, n+1$.

Now,

$$\frac{f(x)}{f'(x)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2 c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5)$$

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (4c_2^3 + 3c_4 - 7c_2 c_3)e_n^4 + O(e_n^5)$$

and

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4c_2(c_3 - c_2^2)e_n^3 + (8c_2^4 - 11c_2^2 c_3 + 6c_2 c_4)e_n^4 + O(e_n^5)] \tag{10}$$

Subsequently,

$$\frac{x_n + y_n}{2} - \alpha = \frac{1}{2}e_n + \frac{1}{2}c_2 e_n^2 + (c_3 - c_2^2)e_n^3 + (2c_2^3 + \frac{3}{2}c_4 - \frac{7}{2}c_2 c_3)e_n^4 + O(e_n^5) \tag{11}$$

$$\frac{3x_n + y_n}{4} - \alpha = \frac{3}{4}e_n + \frac{1}{4}c_2 e_n^2 - \frac{1}{2}(c_3 - c_2^2)e_n^3 + (c_2^3 + \frac{3}{4}c_4 - \frac{7}{2}c_2 c_3)e_n^4 + O(e_n^5) \tag{12}$$

$$\frac{x_n + 3y_n}{4} - \alpha = \frac{1}{4}e_n + \frac{3}{4}c_2 e_n^2 + \frac{3}{4}(c_3 - c_2^2)e_n^3 + (3c_2^3 + \frac{9}{4}c_4 - \frac{21}{4}c_2 c_3)e_n^4 + O(e_n^5) \tag{13}$$

From (11), (12), (13) we obtain

$$f'(\frac{x_n + y_n}{2}) = f'(\alpha)[1 + c_2 e_n + (c_2^2 + \frac{3}{4}c_3)e_n^2 + (-2c_2^3 + \frac{1}{2}c_4 + \frac{7}{2}c_2 c_3)e_n^3 + O(e^4)] \tag{14}$$

$$f'(\frac{3x_n + y_n}{4}) = f'(\alpha)[1 + \frac{3}{2}c_2 e_n + (\frac{1}{2}c_2^2 + \frac{27}{16}c_3)e_n^2 + (-c_2^3 + \frac{27}{16}c_4 + \frac{17}{8}c_2 c_3)e_n^3 + O(e^4)] \tag{15}$$

$$f'(\frac{x_n + 3y_n}{4}) = f'(\alpha)[1 + \frac{1}{2}c_2 e_n + (\frac{3}{2}c_2^2 + \frac{3}{16}c_3)e_n^2 + (-3c_2^3 + \frac{1}{16}c_4 + \frac{33}{8}c_2 c_3)e_n^3 + O(e^4)] \tag{16}$$

Now,

$$2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4}) = 3f'(\alpha)[1 + c_2 e_n + (c_2^2 + c_3)e_n^2 + (c_4 + 3c_2 c_3 - 2c_2^3)e_n^3 + O(e^4)]$$

Therefore

$$\frac{3f(x_n)}{2f'(\frac{3x_n + y_n}{4}) - f'(\frac{x_n + y_n}{2}) + 2f'(\frac{x_n + 3y_n}{4})} = e_n - c_2^2 e_n^3 + O(e_n^4)] \tag{17}$$

Using (17), from equation (8b), we get

$$z_n = \alpha + c_2^2 e_n^3 + O(e^4)$$

Also, using (9), (10), (14), (15) and (16) we obtain

$$f'(y_n)f'(x_n) = f'(\alpha)^2[1 + 2c_2e_n + (2c_2^2 + 3c_3)e_n^2 + 4(c_4 + c_2c_3)e_n^3 + O(e_n^4)]$$

$$f'(x_n)^2 = f'(\alpha)^2[1 + 4c_2e_n + (4c_2^2 + 6c_3)e_n^2 + (8c_4 + 12c_2c_3)e_n^3 + O(e_n^4)]$$

$$f'(x_n)f'(\frac{3x_n + y_n}{4}) = f'(\alpha)^2[1 + \frac{7}{2}c_2e_n + (\frac{7}{2}c_2^2 + \frac{75}{16}c_3)e_n^2 + (\frac{91}{16}c_4 + 10c_2c_3)e_n^3 + O(e^4)]$$

$$f'(x_n)f'(\frac{x_n + y_n}{2}) = f'(\alpha)^2[1 + 3c_2e_n + (3c_2^2 + \frac{15}{4}c_3)e_n^2 + (\frac{9}{2}c_4 + 8c_2c_3)e_n^3 + O(e^4)]$$

$$f'(x_n)f'(\frac{x_n + 3y_n}{4}) = f'(\alpha)^2[1 + \frac{5}{2}c_2e_n + (\frac{5}{2}c_2^2 + \frac{51}{16}c_3)e_n^2 + (\frac{65}{16}c_4 + 6c_2c_3)e_n^3 + O(e^4)]$$

Using the above result in (8c), we get

$$e_{n+1} = c_2^3(c_2^2 - 3c_3)e_n^6 + O(e_n^7)$$

This show that the three step method (i.e. Algorithm RM) has sixth-order convergence.□

Numerical Examples: In this section, we have worked out some examples to illustrate the efficiency of the newly developed three-step iterative method in this paper (i.e Algorithm RM) by comparing the Newton-Raphson method (NRM), the method of Cordero and Torregrosa [5] (ON), the method of Parhi and Gupta [6] (PM) and new algorithm RM, which introduced in this paper. The following stopping criteria are used for compare the methods and to compute the number of iterations:

(1) $|x_{n+1} - x_n| < \varepsilon$ for $\varepsilon = 15^{-15}$
(2) $|f(x_{n+1})| < \varepsilon$ for $\varepsilon = 10^{-15}$

The examples are

$f_1(x) = x^3 + 4x^2 - 10,$ $\alpha = 1.36523001341410.$
$f_2(x) = \sin^2 x - x^2 + 1,$ $\alpha = 1.40449164821534.$
$f_3(x) = x^2 - e^x - 3x + 2,$ $\alpha = 0.25753028543986.$
$f_4(x) = \cos(x) - x$ $\alpha = 0.73908513321516.$
$f_5(x) = (x-1)^3 - 1,$ $\alpha = 2.0.$
$f_6(x) = x^3 - 10,$ $\alpha = 2.15443469003188.$
$f_7(x) = xe^{x2} - \sin^2 x + 3cox + 5,$ $\alpha = -1.20764782713092.$
$f_8(x) = \sqrt{x} - \frac{1}{x} - 3,$ $\alpha = 9.63359556283270.$
$f_9(x) = e^x + x - 20,$ $\alpha = 2.84243895378445.$
$f_{10}(x) = x^3 - x^2 - 1,$ $\alpha = 1.46557123187677.$
$f_{11}(x) = x^3 - 9x^2 + 28x - 30,$ $\alpha = 3.00000000000000.$
$f_{12}(x) = \sin(x) + x\cos(x),$ $\alpha = 2.02875783811043.$
$f_{13}(x) = e^{-x} + \cos(x),$ $\alpha = 1.74613953040801.$
$f_{14}(x) = x^{10} - 1,$ $\alpha = 1.$
$f_{15}(x) = (x-2)^{23} - 1,$ $\alpha = 3.$

Table 3.1: Comparison of various iterative methods by depending on the number of iterations

		Numbers of iterations by			
$f(x)$	x_0	NM	ON[5]	PM[6]	RM
f_1	1.6	5	4	3	3
f_2	1.0	6	4	3	3
f_3	1.5	6	4	3	3
f_4	4.0	30	6	12	5
f_5	1.5	8	6	4	3
f_6	4.0	7	5	4	3
f_7	-1.2	4	3	3	3
f_8	1.3	7	5	6	4
f_9	2.5	6	4	3	3
f_{10}	0.5	13	Div	9	6
f_{11}	0.1	8	6	5	4
f_{12}	3	6	4	4	3
f_{13}	3	8	5	4	3
f_{14}	3.5	18	11	6	5
f_{15}	5	31	20	9	6

Note: From Table 3.1, one can see that our presented method behaves either similar or better than the compared methods

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