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A Modified Series Solution of a Class of Nonlinear Helmholtz Type Equations

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Abstract: In this paper, the Adomian decomposition method is modified for solving some classes of initial and/or boundary value problems of wave, heat and Poisson equations. To modify Adomian series solution, some weighted algorithms are established which used all of conditions of proposed problems simultaneously and effectively. Some examples are considered and the validity of the proposed algorithms are confirmed by the numerical results.

Key words: Weighted algorithm · Helmholtz type equations · Adomian decomposition method · Modified series solution.

INTRODUCTION

Helmholtz-type equations arise naturally in many physical applications related to wave propagation, vibration phenomena and heat transfer. These equations are often used to describe the vibration of a structure, the acoustic cavity problem, the radiation wave, the scattering of a wave, acoustic scattering in fluid-solid problems and the heat conduction in fins [1-14].

In this study, we consider a class of Helmholtz type PDE's having nonlinear term as

$$u_{xx} + u_{yy} + (1 + q(u))u = f(x, y), \quad (1)$$
$$(x, y) \in [0,1] \times [0,1],$$

With the following conditions

$$u(x,0) = p_1(x), \quad 0 < x < 1, \quad (2)$$
$$u(1,t) = p_2(x), \quad 0 < x < 1, \quad (3)$$
$$u(0, y) = q_1(y), \quad 0 < y < 1, \quad (4)$$
$$u(1, y) = q_2(y), \quad 0 < y < 1. \quad (5)$$
$$0 \le u(x, y) \le C, \quad (6)$$

Where q is a known smooth function, P_1, P_2 q_1 q_2 re known L_2 functions and C ia a constant. Existence and uniqueness of solution of Helmholtz type equations and some computational approaches for solving theses kind of problems are discussed in literature [9-14]. Nonlinear partial differential equations are encountered in such various fields as physics, mathematics and engineering. Most nonlinear models of real life problems are still very difficult to solve either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient methods for determining a solution, approximate or exact, analytical or numerical, to the nonlinear models.

The objective of this work is to establish an algorithm based on Adomian decomposition method (ADM) for solving the problem (1)-(6). The ADM has been proved to be effective and reliable for handling differential equations, linear or nonlinear. Unlike the traditional methods, The ADM needs no discritization, linearization, spatial transformation or perturbation. The ADM provide an analytical solution in the form of an infinite convergent power series. A large amount of research works has been devoted to the application of the ADM to a wide class of linear and nonlinear, ordinary or partial differential equations [15-20]. However, when initial and/or boundary conditions have to be imposed, there are still difficulties that cannot be encountered.

Analysis of the Method: Formally in the ADM, we first consider equation (1) in operator form

$$L_{xx}(u) + L_{yy}(u) + N(u) = f(x, y), \quad (7)$$

Where $N(u)$ = $(1+q(u)u$ and L_{xx} and L_{yy} are linear differential operators which defined as

$$L_{xx} = \frac{\partial^2}{\partial x^2}, \quad L_{yy} = \frac{\partial^2}{\partial y^2}. \quad (8)$$

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Applying the inverse operator $L_{xx}^{-1}(.) = \int_0^x\int_0^{x'}(.)dx''dx' - x\int_0^1\int_0^{x'}(.)dx''dx'$ on both sides of equation (7) yields

$$u(x,y) = (1-x)q_1(y) + xq_2(y) + L_{xx}^{-1}(f(x,y) - L_{yy}(u) - N(u)). \quad (9)$$

In the ADM, the solution $u(x,y)$ is given by the series

$$u(x,y) = \sum_{n=0}^{\infty}\hat{u}_n(x,y), \quad (10)$$

and the nonlinear term $N(u)$ are decomposed as

$$N(u) = \sum_{n=0}^{\infty}B_n, \quad (11)$$

Where B_n is the so-called Adomian polynomials which may be found as [15-17]

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{k=0}^{\infty}\lambda^k\hat{u}_k)]_{\lambda=0},\ n \ge 0. \quad (12)$$

Substituting (10) and (11) into (9) gives

$$\sum_{n=0}^{\infty}\hat{u}_n = (1-x)q_1(y) + xq_2(y) + L_{xx}^{-1}[f(x,y) - L_{yy}(\sum_{n=0}^{\infty}\hat{u}_n) - \sum_{n=0}^{\infty}B_n]. \quad (13)$$

The individual terms are obtained using the following recursive scheme [15, 16]

$$\hat{u}_0 = (1-x)q_1(y) + xq_2(y) + L_{xx}^{-1}(f(x,y)) \quad (14)$$

$$\hat{u}_{n+1} = L_{xx}^{-1}[-L_{yy}(\hat{u}_n) - B_n],\ n \ge 0. \quad (15)$$

On the other hand, one may use L_{yy} and its inverse to solve the problem (1)-(6). By using the inverse operator $L_{yy}^{-1}(.) = \int_0^y\int_0^{y'}(.)dy''dy' - y\int_0^1\int_0^{y'}(.)dy''dy',$ and applying it on both sides of equation (7), we have

$$u(x,y) = (1-y)p_1(x) + yp_2(x) + L_{yy}^{-1}(f(x,y) - L_{xx}(u) - N(u)). \quad (16)$$

Using ADM, the solution $u(x, t)$ can be derived as

$$u(x,y) = \sum_{n=0}^{\infty}\tilde{u}_n(x,y), \quad (17)$$

Where

$$\tilde{u}_0 = (1-y)p_1(x) + yp_2(x) + L_{yy}^{-1}(f(x,y)), \quad (18)$$

and

$$\tilde{u}_{n+1} = L_{yy}^{-1}(-L_{xx}(\tilde{u}_n) - B_n),\ n > 0. \quad (19)$$

In (19), B_n is the Adomian polynomial for the function $N(u)$ which obtained with respect to $\tilde{u}_n$, $n \ge 0$. The decomposition series (10) and (17) are generally convergent very rapidly in real physical problems. The convergence analysis of the ADM, applied to various nonlinear equations has been conducted by several authors [18-20]. In this paper, we suppose that $q(u)$ is an analytic function and decomposition series (10) and (17) are convergent.

The boundary value problem (1)-(6) require to solve using all of boundary conditions. But there is no guarantee that the decomposition series (10) would satisfy the conditions (4) and/or (5) and the decomposition series (17) would satisfy the conditions (2) and/or (3).

A Weighted Decomposition Method: Adomian initially established that in PDE problems involving linear operator terms with respect to x; y; z and t, four equations are solved and then a linear combination of these solutions is necessary [15-17]. In this section, the partial series solution of the problem (1)-(6) is derived by a dynamic weighted algorithm based on ADM which is a linear combination of partial series solutions with respect to x and y. The weight coefficients in this linear combination are determined by using all of boundary conditions simultaneously and effectively. We consider that the partial series solution of the problem (1)-(6) is

$$\psi_n(x,y) = \theta_n\phi_n(x,y) + (1-\theta_n)\varphi_n(x,y), \quad (20)$$

Where $\{\theta_n\}_{n\ge 0}$ can be considered any convergent sequence in [0,1] and

$$\phi_n(x,y) = \sum_{k=0}^{n-1}\hat{u}_k(x,y), \quad (21)$$

$$\varphi_n(x,y) = \sum_{k=0}^{n-1}\tilde{u}_k(x,y). \quad (22)$$

Following theorem shows that the optimum values of the weight coefficients θ_n for each $n \ge 0$ may be determined with respect to the boundary conditions.

Theorem: Suppose that for x,y∈(0,1): $p_1(x)$, $p_2(x)$ $q_1(y)$ and $q_2(y)$ are L_2 functions and,

$$x_{1n} = \| \phi_n(x,0) - p_1(x) \|,$$
$$x_{2n} = \| \phi_n(x,1) - p_2(x) \|,$$
$$\lambda_{1n} = \| \phi_n(0,y) - q_1(y) \|,$$
$$\lambda_{2n} = \| \phi_n(0,y) - p_2(y) \|,$$

Where $\|\cdot\|$ denotes the L_2-norm. Then the optimum values of θ_n n (20) is

$$\theta_n = \frac{\lambda_{1n}^2 + \lambda_{1n}^2}{\lambda_{1n}^2 + \lambda_{1n}^2 + \chi_{1n}^2 + \chi_{2n}^2}, \quad n \geq 0. \tag{23}$$

Proof: Consider the residual functional J_n as follows

$$J_n = \| \psi_n(0,y) - q_1(y) \|^2 + \| \psi_n(1,y) - q_2(y) \|^2 + \| \psi_n(x,0) - p_1(x) \|^2 + \| \psi_n(x,1) - p_2(x) \|^2.$$

Using (20) yields

$$J_n = \| \theta_n\phi_n(0,y) + (1-\theta_n)\varphi_n(0,y) - q_1(y) \|^2 + \| \theta_n\phi_n(1,y) + (1-\theta_n)\varphi_n(1,y) - q_2(y) \|^2 + \| \theta_n\phi_n(x,0) + (1-\theta_n)\varphi_n(x,0) - p_1(x) \|^2 + \| \theta_n\phi_n(x,1) + (1-\theta_n)\varphi_n(x,1) - p_2(x) \|^2.$$

According to the definitions of $\phi_n(x,t)$ and $\varphi_n(x,t)$ we see that

$$\phi_n(0,y) = \hat{u}_n(0,y) = q_1(y),$$
$$\phi_n(1,y) = \hat{u}_n(1,y) = q_2(y),$$
$$\varphi_n(x,0) = \tilde{u}_n(x,0) = p_1(x),$$
$$\varphi_n(x,1) = \tilde{u}_n(x,1) = p_2(x).$$

Therefore,

$$J_n = \theta_n^2(\| \phi_n(x,0) - p_1(x) \|^2 + \| \phi_n(x,1) - p_2(x) \|^2) + (1-\theta_n)^2(\| \varphi(0,y) - q_1(y) \|^2 + \| \varphi(1,y) - q_2(y) \|^2) = \theta_n^2(\chi_{1n}^2 + \chi_{2n}^2) + (1-\theta_n)^2(\lambda_{1n}^2 + \lambda_{1n}^2)$$

It is natural to choose the value of θ_n such that this residual is minimized. Since J_n is a quadratic function of θ_n, we can find the minimum by Differentiating J_n with respect to θ_n and setting the result to zero. This yields

$$\theta_n = \frac{\lambda_{1n}^2 + \lambda_{1n}^2}{\lambda_{1n}^2 + \lambda_{1n}^2 + \chi_{1n}^2 + \chi_{2n}^2}, \quad n \geq 0,$$

and this complete the proof of theorem.

Numerical Experiments: In order to present the performance of the numerical method proposed, we illustrate the numerical results obtained using the weighted algorithm introduced in section 3 for solving the two test problems.

Example 1: Consider following nonlinear helmholtz type problem

$$u_{xx} + u_{yy} + (1 - (9u + 2u^2))u = 0, \quad (x,y) \in [0,1]\times[0,1],$$

$$u(x,0) = \frac{1}{\sqrt{8}\cos(\frac{\sqrt{2}}{2}x) + 3}, \quad 0 < x < 1,$$
$$u(1,x) = \frac{1}{\sqrt{8}\cos(\frac{\sqrt{2}}{2}(x+1)) + 3}, \quad 0 < x < 1,$$
$$u(0,y) = \frac{1}{\sqrt{8}\cos(\frac{\sqrt{2}}{2}y) + 3}, \quad 0 < y < 1,$$
$$u(1,y) = \frac{1}{\sqrt{8}\cos(\frac{\sqrt{2}}{2}(y+1)) + 3}, \quad 0 < y < 1.$$

The exact solution of this problem can be derived as

$$u(x,y) = \frac{1}{\sqrt{8}\cos(\frac{\sqrt{2}}{2}(x+y)) + 3}. \tag{24}$$

Table 1 shows the decomposition solution using Ψ_5 (x,y), exact solution $u(x,y)$ and the absolute errors between them at some points.

Example 2: Consider following nonlinear boundary value problem

$$u_{xx} + u_{yy} + (1 - 2u^3)u = 0, \quad (x,y) \in [0,1]\times[0,1],$$

$$u(x,0) = \frac{1}{\cos\frac{1}{2}x}, \quad 0 < x < 1,$$
$$u(1,x) = \frac{1}{\cos\frac{1}{2}(x+\sqrt{3})}, \quad 0 < x < 1,$$
$$u(0,y) = \frac{1}{\cos\frac{\sqrt{3}}{2}y}, \quad 0 < y < 1,$$
$$u(1,y) = \frac{1}{\cos\frac{1}{2}(1+\sqrt{3}y)}, \quad 0 < y < 1.$$

The exact solution of this problem is

$$u(x,y) = \frac{1}{\cos(\frac{1}{2}(x+\sqrt{3}y))}. \tag{25}$$

Table 1: The comparison between exact and modified decomposition solutions

x	y	Exact	Ψ_5	Absolute errors
0.2	0.3	0.176882	0.176892	4.87892E-6
	0.6	0.185604	0.185607	8.52835E-6
	0.9	0.199398	0.199405	9.08611E-6
0.4	0.3	0.182184	0.182191	4.49177E-6
	0.6	0.194164	0.194165	9.56048E-6
	0.9	0.212079	0.212087	3.61354E-6
0.6	0.3	0.189583	0.189585	4.35648E-6
	0.6	0.205346	0.205354	8.77005E-6
	0.9	0.228253	0.228257	5.56102E-6

Table 2: The comparison between exact and modified decomposition solutions

x	y	Exact	Ψ_5	Absolute errors
0.2	0.3	1.05077	1.05078	8.17711E-6
	0.6	1.15514	1.15515	6.58399E-6
	0.9	1.34972	1.34973	7.77667E-6
0.4	0.3	1.09138	1.09139	4.40764E-6
	0.6	1.23245	1.23247	3.85619E-6
	0.9	1.49222	1.49224	3.43126E-6
0.6	0.3	1.14718	1.14719	3.74306E-6
	0.6	1.33514	1.33515	2.69698E-6
	0.9	1.68721	1.68723	3.22077E-6

In Table 2, we show the comparison of the decomposition solution using Ψ_6 (x,y) and the exact solution $u(x,y)$ at some points.

CONCLUSION

In this study a modification of Adomian decomposition method (ADM) is developed to solve a class of nonlinear Helmholtz type equations with Dirichlet boundary conditions. For this end a weighted algorithm based on ADM is established with respect to the boundary conditions. In spite of classical decomposition methods, the proposed method applies all boundary conditions simultaneously and effectively to represent the solution. This method may be use to solve Neumann and mixed boundary value problems.

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