World Applied Sciences Journal 11 (10): 1303-1306, 2010
ISSN 1818-4952

α-Nonexpansive Mappings on CAT(0) Spaces

Mehdi Asadi, S. Mansour Vaezpour and Hossein Soleimani

Department of Mathematics,
Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract: In this paper the α-nonexpansive map is introduced and some properties of their fixed points are verified.

AMS Subject Classification: 05C05; 54H25
Key words: CAT (0) spaces · Hyperbolic spaces · Fixed Points

INTRODUCTION

Let (X,d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map *c* from a closed interval $[0, \ell] \subseteq R$ to X such that $c(0) = x, c(\ell) = y$ and $d(c(t),c(t_0)) = |t - t_0|$ for all $t,t_0 \in [0,\ell]$. In particular, c is an isometry and $d(x,y) = \ell$. The image α of c is called a geodesic (or metric) segment joining *x* and *y* When it is unique, this geodesic is denoted by [x,y]. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in *X* (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for a geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane E^2 such that $d_{E^2}(\bar{x}_i,\bar{y}_j) = d(x_i,y_j)$ for $i, j \in \{1,2,3\}$.

A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

"Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x,y \in \Delta$ and all comparison points $x, y \in \bar{\Delta}$,

$$d(x,y) \le d_{E^2}(\bar{x},\bar{y})."$$

In the following we recall some useful lemmas and theorem. For more details refer [1-6].

Lemma 1.1: ([7, Lemma 2.5]) Let (X, d) be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2,$$

for all x, y, z, $\in$ X and $t \in [0,1]$.

In particular by Lemma 1.1 we have

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, z)^2 \le \frac{1}{2}d(x,z)^2 + \frac{1}{2}d(y,z)^2 - \frac{1}{4}d(x,y)^2,$$

for all $x, y, z \in X$, which is called (CN) inequality of Bruhat-Tits, as it was shown in [8]. In fact (cf. [9], pp: 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

In the sequel, we let $n \in N$, $z_1 = x$ and $z_n = y$ until Definition 1.4.

Lemma 1.2: ([10]) Let (X,d) be a CAT(0) space. Then

1. Let $x, y \in X$, $x \ne y$ and $z_i, z'_i \in [x,y]$ such that $d(x,z_i) = d(x, z'_i)$ for all $1 \le i \le n$. Then $z_i = z'_i$ for $1 \le i \le n$.
2. Let $x,y \in X$, then for each $\alpha = (\alpha_1,\ldots,\alpha_n) \in [0,1]^n$ with $\sum_{i=1}^{n} \alpha_i = 1$ there exist unique points $z_1,\ldots, z_n \in [x,y]$ and unique point $z \in [x,y]$ such that $d(z, z_i) = \alpha_i \ell$ for $1 \le i \le n$.

Corresponding Author: Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
Fax: +98-21-44867197, E-mail: masadi@azu.ac.ir

Notation: By the point z_α, we mean the unique point

$$z_\alpha = \alpha_1 z_1 \oplus \ldots \oplus \alpha_n z_n$$

Where $\alpha = (\alpha_1,\ldots,\alpha_n) \in [0,1]^n$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $z_i \in X$ for $1 \le i \le n$. Also z_α, can be written as

$z_\alpha = (1-\alpha_n)z' \oplus \alpha_n z_n$,

Where $z' = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \ldots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}$.

Theorem 1.3: ([10]) Let (X,d) be a CAT(0) space, let $x,y \in X$, such that $x \neq y$. Then

1. $[x,y] = \{z_\alpha \mid \alpha \in [0,1]^n, \sum_{i=1}^{n} \alpha_i = 1\}$.

2. For all $z \in X$ the following holds: ($\exists z_1,\cdots,z_n \in [x,y]$ such that $\sum_{i=1}^{n} d(z,z_i) = d(x,y)$) if and only if $z \in [x,y]$.

3. The mapping $f:[0,1]^n \to [x,y], f(\alpha) = z_\alpha$ is continuous and bijective

4. $d(z_\alpha, z) \le \sum_{i=1}^{n} \alpha_i d(z_i, z) \le \max\{d(z_i,z) : 1 \le i \le n\}$,

5. $d(z_\alpha, z)^2 \le \sum_{i=1}^{n} \alpha_i d(z_i, z)^2 \le \max\{d(z_i,z)^2 : 1 \le i \le n\}$,

6. $d(z_\alpha, z'_\beta) \le \sum_{i,j=1}^{n} \alpha_i \beta_j d(z_i, z'_j) \le \max\{d(z_i,z'_j) : 1 \le i,j \le n\}$,

for $\alpha = (\alpha_1,\ldots,\alpha_n), \beta = (\beta_1,\ldots,\beta_n) \in [0,1]^n$ with $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$

and $z, z_i, z'_i \in X$ for $1 \le i \le n$ which

$$z_\alpha = \alpha_1 z_1 \oplus \cdots \oplus \alpha_n z_n, \quad z'_\beta = \beta_1 z'_1 \oplus \cdots \oplus \beta_n z'_n.$$

In the sequel we used this notation $T_\alpha = -\alpha_1 T_1 \oplus \ldots \oplus \alpha_n T_n$ where $T_1,\ldots,T_n$ are maps on X, Such that $T_i u \in [x,y]$ for all $u \in X$, $1 \le i \le n$ and $\alpha = (\alpha_1,..,\alpha_n) \in [0,1]^n$ a multiindex satisfying

$$\sum_{i=1}^{n} \alpha_i = 1.$$

Definition 1.4: ([11]) Let $\alpha = (\alpha_1,..,\alpha_n) \in [0,1]^n$ be a multiindex satisfying $\sum_{i=1}^{n} \alpha_i = 1$ The maps $T_1,\ldots,T_n$ on X are said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i d(T_i x, T_i y) \le d(x,y), \tag{1.1}$$

for all $x, y \in X$.

For a simple case of the definition 1.4 we can consider the following definition for a map.

Definition 1.5: Let $\alpha = (\alpha_1,..,\alpha_n) \in [0,1]^n$ be a multiindex satisfying $\sum_{i=1}^{n} \alpha_i = 1$ A mapping $T:X \to X$ is said to be α-nonexpansive if

$$\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) \le d(x,y)$$

for all $x, y \in X$.

Remark 1.6: The following observations are immediate.

1. The condition (1.1) implies that $T_1,\ldots,T_n$ are α-nonexpansive and the mapping T_α is nonexpansive when $T_1,\ldots,T_n$ are nonexpansive in the hyperbolic spaces (X,d) because by the Theorem 1.3 we have $d(T_\alpha x, T_\alpha y) = d(\alpha_1 T_1 x \oplus \cdots \oplus \alpha_n T_n x, \alpha_1 T_1 y \oplus \cdots \oplus \alpha_n T_n y)$
$$\le \sum_{i=1}^{n} \alpha_i d(T_i x, T_i y) \le d(x,y),$$
for every $x, y \in X$. However, the condition (1.1) is stronger.
2. From α-nonexpansiveness of T_α we have $\alpha_i, d(T_i x, T_i y) \le d(x,y)$ for all $1 \le i \le n$ so, if α_j for some $1 \le j \le n$ then T_j is nonexpansive, in which case $T_\alpha = T_j$ and if $\alpha_j \neq 0$ for $1 \le i \le n$ then T_i is Lipschitz thus, it is uniformly continuous.
3. All nonexpansive mappings satisfy (1.1).

Example 1.7: Let $X = \ell^1$ with $d(x,y) = \|x-y\|_{\ell^1}$ and B be its unit ball. Define the mapping $T : B \to B$ by

$$Tx = T(x_1, x_2, x_3, \cdots) = (x_2^2, ax_3, x_4, \cdots)$$

where $a = \frac{\sqrt{17}-1}{4}$. T is α-nonexpansive with $\alpha = (\alpha_1, \alpha_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$. The Lipschitz constant $k(T)$ is 2 and $k(T)^2 = 2\alpha^2 > 1$. For further iterations,

$$T^n x = (a^2 x_{n+1}^2, ax_{n+2}, x_{n+3}, \cdots), \quad n \geq 3.$$

All have the same Lipschitz constant as $T^2, k(T^n) = k(T^2) > 1$. For more details refer [11].

MAIN RESULTS

Lemma 2.1: Let (X,d) be a complete CAT(0) space. If T,S be two self map on X with $F(T) \cap F(S) \neq \phi$ and $U = (1-t)T \oplus tS$ a self map on X for every $t \in [0,1]$. Then

(I) There exists a continuous map T' on X such that $F(T') = F(T) \cap F(S)$

(ii) $d(T'x, Ux) \leq (1-t)d(T'x, Tx) + td(T'x, Sx)$, for every $x \in X$

(iii) $d(Ux, Uy) \leq (1-t)d(Tx, Ty) + td(Sx, Sy)$, for every $x,y \in X$ and $F(T') \subseteq F(U)$.

(iv) Further, if T and S be nonexpansive self maps on X, then U is nonexpansive too and $F(U) \subseteq F(T')$

Proof: (i), (ii) and (iii) can be easily proved, so we prove (iv). Let $x \in F(U)$ and $y \in F(T')$ so

$$d^2(x,y) = d^2(Ux,y)$$

$$\leq (1-t)d^2(Tx,y) + td^2(Sx,y) - t(1-t)d^2(Tx,Sx)$$

$$= (1-t)d^2(Tx,Ty) + td^2(Sx,Sy) - t(1-t)d^2(Tx, Sx)$$

$$\leq d^2(x,y) - t(1-t)d^2(Tx, Sx),$$

therefore $Sx = Tx = Ux = x$ namely $x \in F(T) \cap F(S) = F(T')$, which it completes (iv).□

Theorem 2.2: Let $(X\ d)$, be a complete CAT(0) space. If $T_\alpha = \alpha_1 T_1 \oplus ... \oplus \alpha_n T_n$ where $T_1, ..., T_n$ are selfmaps on X for $1 \leq i \leq n$, then

1. There exists a continuous map T' on X such that $F(T') = \bigcap_{i=1}^{n} F(T_i)$.
2. $F(T') \subseteq F(T_\alpha)$ and $d(T'x, T_\alpha x) \leq \sum_{i=1}^{n} \alpha_i d(T'x, T_i x)$, for every $x \in X$.
3. If T_i be nonexpansive selfmaps on X for $1 \leq i \leq n$, then T_α isnonexpansive too and $F(T_\alpha) \subseteq F(T')$.

Proof: Let $x = T'x$ so $x = T_i x$ for $1 \leq i \leq n$, therefore,

$$T_\alpha x = \alpha_1 T_1 x \oplus \cdots \oplus \alpha_n T_n x = \alpha_1 x \oplus \cdots \oplus \alpha_n x = x,$$

because $d(x, \alpha_1 x \oplus ... \oplus \alpha_n x) = 0$, thus $x = T_\alpha x$. This completes (2). To prove (3) we have

$$d(T_\alpha x, T_\alpha y) \leq \sum_{i=1}^{n} \alpha_i d(T_i x, T_i y) \leq \sum_{i=1}^{n} \alpha_i d(x,y) \leq d(x,y),$$

So T_α is nonexpansive.

We shall show that $F(T_\alpha) \subseteq F(T')$ holds for $n \geq 2$. It is true for $n = 2$. Now

Consider $T_\alpha = (1-\alpha_n)U \oplus \alpha_n T_n$ where $U = \frac{\alpha_1}{1-\alpha_n} T_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} T_{n-1}$.

Let $y \in F(T')$ and $x \in F(T_\alpha)$. Since $y \in F(T') \subseteq F(U)$ and $y \in F(T') \subseteq F(T_i)$

for $1 \leq i \leq n$, thus $y = T'y = T_i y$ so $Uy = \frac{\alpha_1}{1-\alpha_n} y \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} y$.

Since $d(y,Uy) = \frac{\alpha_1}{1-\alpha_n} d(y,y) = 0$ hence $y \in F(U)$ Now by induction $F(U) \subseteq F(T')$. Now since U is nonexpansive and

$$d^2(x,y) = d^2(T_\alpha x, y),$$

$$\leq (1-\alpha_n)d^2(Ux,y) + \alpha_n d^2(T_n x, y) - \alpha_n(1-a_n)d^2(Ux, T_n x),$$

$$= (1-\alpha_n)d^2(Ux,Uy) + \alpha_n d^2(T_n x, T_n y) - \alpha_n(1-\alpha_n)d^2(Ux, T_n x),$$
$$\leq d^2(x,y) - \alpha_n(1-\alpha_n)d^2(Ux, T_n x),$$

So $Ux = T_n x$, $T_\alpha x = x$ and

$$x = T_a x = (1-a_n)Ux \oplus a_n T_n x = (1-a_n)Ux \oplus a_n Ux = Ux,$$

then $Ux = x = T_n x$ and so $x \in F(U)$

Since x ∈ F(T_i) for 1 ≤ I ≤ n – 1 so $x \in \bigcap_{i=1}^{n} F(T_i) = F(T')$ thus T' x = x .

Lemma 2.3: Let (X,d) be a CAT(0) space, F_α = F(T_α) and

$$\Im = \{F(T) \mid T : X \to X, F(t) \neq \phi, F(T) closed\} \cup \{\phi, X\},$$

then

a) If $F_\alpha \in \Im$ for every $\alpha \in I$, then $\bigcap_\alpha F_\alpha \in \Im$.

b) If $F_i \in \Im$ for 1 ≤ i ≤ n, then $\bigcup_{i=1}^{n} F_i \in \Im$.

Proof: If $\bigcap_\alpha F_\alpha = \phi$ then $\bigcap_\alpha F_\alpha \in \Im$. Otherwise $\bigcap_\alpha F_\alpha$ is nonempty and closed.
So by [1, Theorem 2.1] there exists continuous map T: X→ X such that $F(T) = \overline{\bigcap_\alpha F_\alpha} = \bigcap_\alpha F_\alpha$ so $\bigcap_\alpha F_\alpha \in \Im$. This completes (a).□

Lemma 2.4: With assumptions of Lemma 2.3, if $\tau = \{F \mid F^C \in \Im\}$, Then τ is a topology on X

Proof: By Lemma 2.3, τ is a topology on X□.

REFERENCES

1. Chaoha, P. and A. Phon-on, 2006. A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl., 320: 983-987.
2. Leustean, L., 2007. A quadratic rate of asymptotic regularity for CAT(0)-spaces, J. Math. Anal. Appl., 325: 386399.
3. Nanjaras, B. and B. Panyanaka and W. Phuengrattana, 2010. Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansivemappings in CAT(0) spaces, Nonlinear Analysis: Hybrid Systems, 4: 25-31.
4. Razani, A. and H. Salahifard, 2010, Invariant approximation for CAT(0) spaces, Nonlinear Analysis: Theory, Methods and Applications, 72: 2421-2425.
5. Saejung, S., 2010. Halpern's Iteration in CAT(0) Spaces, Fixed point theory and applications, Article ID 471781, pp: 13.
6. Suzuki, T., 2008. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl., 340: 1088-1095.
7. Dhompongsa, S. and B. Panyanak, 2008. On Δ-convergence theorems in CAT(0) spaces, Comp. and Math. with Appl., 56: 2572-2579.
8. Bruhat, F. and J. Tits, 1972. Groupes ŕeductifs sur un corps local. I. Données radicielles valuées, Inst. Hautes Études Sci. Publ. Math., pp: 415-251.
9. Bridson, M. and A. Haeiger, 1999. Metric Spaces of Non-Positive Curvature, Springer-Verlag, Berlin, Heidelberg.
10. Asadi, M., S.M. Vaezpour and H. Soleimani, 0000. Some Results on Fixed Points and Approximation for a New Class of Mappings in CAT(0) Spaces, Submitted.
11. Goebel, K. and W.A. Kirk, 2008. Some problems in metric fixed point theory, J. Fixed Point Theory and Applications, 4: 13-25.