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Dual Spacelike Biharmonic Curves with Timelike Binormal in the Dual Lorentzian Space $\mathbb{D}_1^3$

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Abstract: In this paper, we study dual spacelike biharmonic curves with timelike binormal in dual Lorentzian space $\mathbb{D}_1^3$ We characterize curvature and torsion of dual spacelike biharmonic curves with timelike binormal in terms of dual Frenet frame in dual Lorentzian space $\mathbb{D}_1^3$.

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INTRODUCTION

A smooth map $\phi:(N,h)\to(M,g)$ between manifolds is said to be harmonic if it is a critical point of the energy functional:

$$E(\phi)=\int_N e(\phi)dv_h,$$

Where $2e(\phi)=|d\phi|^2$. The Euler-Lagrange equation of this variational problem is given by

$$\tau(\phi)=0,$$

Where the vector field $\tau(\phi)$ along ϕ is defined by

$$\tau(\phi)=\mathrm{trace}_h(\nabla d\phi).$$

The vector field $\tau(\phi)$ is called the tension field of ϕ. The study on harmonic maps is one of the central topics in modern differential geometry [1].

A Jacobi field along a harmonic map $\phi:(N,h)$ between manifolds is a vector field along ϕ which is in the kernel of the second variation of the energy functional (Jacobi operator of ϕ). Equivalently, it is tangent to a variation of ϕ for which the tension field remains zero to first order.

A smooth map ϕ : (N,h) between Riemannian manifolds is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi)=\int_N \frac{1}{2}\left|\tau(\phi)\right|^2 dv_h.$$

Clearly, harmonic maps are biharmonic. A biharmonic map is said to be proper if it is not harmonic [2,3]. A smooth map ϕ : (N,h) is biharmonic if and only if its tension field $\tau(\phi)$ is in the kernel of the Jacobi operator of ϕ i.e.

$$\tau_2(\phi)=-J^{\phi}\left(\tau(\phi)\right)=-\Delta\tau(\phi)-\mathrm{trace}R^M\left(d\phi,\tau(\phi)\right)d\phi, \quad (1.1)$$

Where J^{ϕ} is the Jacobi operator of ϕ. The equation $\tau_2(\phi)$ is called the biharmonic equation.

E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [4]. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the points of dual unit sphere S^2 and the directed lines in R^3.

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities [5].

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In this paper, we study dual spacelike biharmonic curves with timelike binormal in dual Lorentzian space D_1^3 We characterize curvature and torsion of dualspacelike biharmonic curves with timelike binormal in terms of dual Frenet frame in dual Lorentzian space D_1^3.

Preliminaries: In the Euclidean 3-Space E^3, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines E^3 are in one to one correspondence with the points of the dual unit sphere D^3.

W.K. Clifford, in [6], introduced dual numbers with the set

$$D = \{\hat{x} = x + \varepsilon x^* : x, x^* \in R\}.$$

The symbol ε designates the dual unit with the property $\varepsilon^2=0$ for $\varepsilon \neq 0$. Thereafter, A good amount of reserach work has been done on dual numbers, dual functions and as well as dual curves [7]. Then, dual angle is introduced, which is defined as $\hat{\theta} = \theta + \varepsilon\theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. The set D of dual numbers is a commutative ring with the operations (+) and (.). The set $D^3 = D \times D \times D = \{\hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^*, \varphi, \varphi^* \in E^3\}$ is a module over the ring D, [8].

Let us denote

$$\hat{a} = a + \varepsilon a^* = (a_1, a_2, a_3) + \varepsilon(a_1^*, a_2^*, a_3^*)$$

and

$$\hat{b} = b + \varepsilon b^* = (b_1, b_2, b_3) + \varepsilon(b_1^*, b_2^*, b_3^*).$$

The Lorentzian inner product of $\hat{a}$ and $\hat{b}$ defined by

$$\langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle + \varepsilon(\langle a, b^* \rangle + \langle a^*, b \rangle).$$

We call the dual space D^3 together with Lorentzian inner product as dual Lorentzian space and show by D_1^3. We call theelements of D_1^3 the dual vectors. For $\hat{\varphi} \neq 0$, the norm $\|\hat{\varphi}\|$ of is defined by $\|\hat{\varphi}\| = \sqrt{|\langle \hat{\varphi}, \hat{\varphi} \rangle|}$. A dual vector $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is called dual space-like vector if $\langle \hat{\varphi}, \hat{\varphi} \rangle > 0$ or $\varepsilon = 0$, dual time-like vector if $\langle \hat{\varphi}, \hat{\varphi} \rangle < 0$, and dual null (light-like) vector if $\langle \hat{\varphi}, \hat{\varphi} \rangle = 0$ for $\hat{\varphi} \neq 0$. Therefore, an arbitrary dual curve, which is a differentiable mapping onto D_1^3, can locally be dual space-like, dual time-like or dual null, if its velocity vectoris respectively, dual space-like, dual time-like or dual null.

Besides, for the dual vectors $\hat{a}, \hat{b} \in D_1^3$ Lorentzian vector product of dual vectors is defined by

$$\hat{a} \times \hat{b} = a \times b + \xi(a^* \times b + a \times b^*),$$

Where $a \times b$ is the classical Lorentzian cross product.

Dual Spacelike Biharmonic Curves in the Dual Lorentzian Space D_1^3

Let $\hat{\gamma} = \gamma + \varepsilon\gamma^* : I \subset R \to D_1^3$ be a C^4 dual spacelike biharmonic curves with timelike binormal by the arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{t}$ is defined and the principal normal is $\hat{n} = \frac{1}{\hat{\kappa}} \nabla_{\hat{t}} \hat{t}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \|\nabla_{\hat{t}} \hat{t}\| = \kappa + \varepsilon\kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{b} = \hat{t} \times \hat{n}$. Hence, the triple $\{\hat{t}, \hat{n}, \hat{b}\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{bmatrix} \nabla_{\hat{t}} \hat{t} \\ \nabla_{\hat{t}} \hat{n} \\ \nabla_{\hat{t}} \hat{b} \end{bmatrix} = \begin{bmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & \hat{\tau} & 0 \end{bmatrix} \begin{bmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{bmatrix}, \tag{3.1}$$

Where $\hat{\tau} = \tau + \varepsilon\tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual.

Theorem 3.1:
Let $\hat{\gamma} : I \to D_1^3$ be a non-geodesic spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike biharmonic curves with timelike binormal curve if and only if

$$\begin{aligned} &\kappa = \text{constant and } \kappa^* = \text{constant}, \\ &\tau = \text{constant and } \tau^* = \text{constant}, \\ &\kappa^2 - \tau^2 + \varepsilon\left(2\kappa\kappa^* - 2\tau\tau^*\right) = 0. \end{aligned} \tag{3.2}$$

Proof: From (1.1), we get the biharmonic equation of $\hat{\gamma}$

$$\tau_2(\hat{\gamma}) = \nabla_{\hat{t}}^3 \hat{t} - R\left(\hat{t}, \nabla_{\hat{t}} \hat{t}\right)\hat{t} = 0 \tag{3.3}$$

Next, using the Frenet equations (3.1) we obtain

$$\nabla_{\hat{t}}^3 \hat{t} = \left(-3\hat{\kappa}\hat{\kappa}'\right)\hat{t} + \left(\hat{\kappa}'' - \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa}\right)\hat{n} + \left(2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa}\right)\hat{b}. \tag{3.4}$$

Thus, (3.3) and (3.4) imply

$$\left(-3\hat{\kappa}\hat{\kappa}'\right)\hat{t} + \left(\hat{\kappa}'' - \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa}\right)\hat{n} + \left(2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa}\right)\hat{b} - \hat{\kappa}R\left(\hat{t}, \hat{n}\right)\hat{t} = 0. \tag{3.5}$$

In D^3, the Riemannian curvature is zero, we have

$$\left(-3\hat{\kappa}\hat{\kappa}'\right)\hat{t} + \left(\hat{\kappa}'' - \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa}\right)\hat{n} + \left(2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa}\right)\hat{b} = 0 \tag{3.6}$$

From Frenet frame, we have

$$\hat{\kappa}\hat{\kappa}' = 0. \tag{3.7}$$

Also, from (3.7) we get

$$\hat{\kappa} = \text{constant}. \tag{3.8}$$

Using $\hat{\kappa} = \kappa + \varepsilon\kappa^*$, we get

$$\kappa = \text{constant and } \kappa^* = \text{constant}. \tag{3.9}$$

From (3.6), we get

$$\hat{\kappa}'' - \hat{\kappa}^3 + \hat{\tau}^2\hat{\kappa} = 0,$$
$$2\hat{\kappa}'\hat{\tau} + \hat{\tau}'\hat{\kappa} = 0.$$

Also, using $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ and $\hat{\tau} = \tau + \varepsilon\tau^*$ we obtain

$$\kappa^2 - \tau^2 + \varepsilon\left(2\kappa\kappa^* - 2\tau\tau^*\right) = 0, \tag{3.10}$$

$$\tau' + \varepsilon\tau^{*\prime} = 0. \tag{3.11}$$

A direct computation using (3.11), yields

$$\tau = \text{constant and } \tau^* = \text{constant}. \tag{3.12}$$

Therefore, we obtain (3.2)

Corollary 3.2: Let $\hat{\gamma}: I \to D_1^3$ be a non-geodesic spacelike biharmonic curves with timelike binormal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike biharmonic curves with timelike binormal if and only if

$$\kappa^2 = \tau^2 = \text{constant} \neq 0, \tag{3.13}$$

$$\kappa\kappa^* = \tau\tau^* = \text{constant} \neq 0. \tag{3.14}$$

Proof: Using (3.10), we have (3.13) and (3.14).

Corollary 3.3: Let $\hat{\gamma}: I \to D_1^3$ be a non-geodesic spacelike biharmonic curves with timelike binormal parametrized by arc length. Then, $\hat{\gamma}$ is a dual helix.

Corollary 3.4: Let $\hat{\gamma}: I \to D_1^3$ be a non-geodesic spacelike biharmonic curves with timelike binormal parametrized by arc length. If $\tau = 0$ then $\hat{\gamma}$ is not biharmonic.

Proof: Above, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. If $\tau = 0$ then this is a contradiction.

CONCLUSION

In this paper, we study dual spacelike biharmonic curves with timelike binormal in dual Lorentzian space D_1^3 We characterize curvature and torsion of dual spacelike biharmonic curves with timelike binormal in terms of dual Frenet frame in dual Lorentzian space D_1^3.

In recent years, dual numbers have been applied to study the motion of a line in space; they seem even to be most appropiate way for this end and they have triggered use of dual numbers in kinematical problems. For instance [8-11]. We hope these results will be helpful to the mathematicians who are specialized on mathematical modelling.

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