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Numerical Solution of Optimal Control Problems Governed by Integro-Differential Equations via a Hybrid Iterative Scheme

Akbar H. Borzabadi and Mojtaba Azizsefat

Department of Applied Mathematics, Damghan University, Damghan, Iran

Abstract: This paper presents a hybrid iterative scheme based on perturbation and parameterization approaches for solving optimal control problems governed by integro-differential equations. The control and state functions are considered as polynomials with unknown coefficients. This converts the problem to nonlinear optimization problems in any iteration. The numerical examples are proposed for showing good ability of the given approach in finding approximate solutions for optimal control problems governed by integro-differential equations.

Key words: Optimal control problem · Integro-differential equation · Perturbation method · Parameterization method · Approximation

INTRODUCTION

Solutions for optimal control problems, except for the simplest cases, are usually carried out numerically. Therefore numerical methods and algorithms for solving optimal control problems have evolved significantly. An overview of numerical methods for solving optimal control problems described by ODE and integral equations can be found in [1].

The current work intends to combine the methods of parametrization [2-4] and perturbation method [5, 6], both are successful methods for solving some classes of optimal control problems and differential-integral equations, respectively, to provide a new numerical scheme for detecting approximate optimal control of systems governed by a class of integro-differential equations described by the following minimization problem:

$$Minimize\ J(x,u)=\int_0^T f_0(t,x(t),u(t))\,dt, \tag{1}$$

Subject to

$$x^{(n)}(t)=y(t)+\int_0^t k_1(s,t,x(s),\ldots,x^{(n)}(s),u(s))ds+\int_0^T k_2(s,t,x(s),\ldots,x^{(n)}(s),u(s))ds, \tag{2}$$

With initial and final conditions

$$x^{(r)}(0)=x^{(r)}_0, x^{(r)}(T)=x^{(r)}_T,\ 0\le r\le n-1, \tag{3}$$

Where $k_1,k_2 \in L^2([0,T]\times[0,T]\times\mathbb{R}^{n+2})$, r is order of integro-differential equation and $f\in C([0,T]\times\mathbb{R}\times\mathbb{R})$. Thereafter, without loss of generality we suppose that $T=1$.

Required Perturbation Scheme: Excellent idea of coupling the traditional perturbation method and homotopy in topology, i.e. homotopy perturbation method (HPM), which was introduced first by J. Huan He in 1999 [7], has been extended and improved by many scientist and engineers as an applicable tool for obtaining approximate solution in a wide range of problems in applied mathematics. Because this method continuously deforms the difficult problem under the study into a simple problem which is easier to solve. Especially, this method has been applied for solving an extensive class of integral equations and we address only [5, 6, 8-10].

To illustrate the basic ideas of this method, we consider the following equation

$$A(v)-f(r)=0,\ r\in\Omega, \tag{3}$$

with the boundary conditions

$$B(v,\frac{\partial v}{\partial n})=0,\quad r\in\Gamma, \tag{4}$$

Corresponding Author: Akbar H. Borzabadi, Department of Applied Mathematics, Damghan University, Damghan, Iran.
E-mail: borzabadi@du.ac.ir. & akbar.h.borzabadi@gmail.com.

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear operator. Therefore, (3) can be rewritten as follows

$$L(v) + N(v) - f(r) = 0. \quad (5)$$

By the homotopy technique, we construct a homotopy, $V(r, p) = \Omega\times[0,1]\rightarrow\mathbb{R}$, which satisfies

$$H(V, p) = (1 - p)(L(V) - L(v_0)) + p(A(V) - f(r)) = 0, p\in[0,1], r \in \Omega \quad (6)$$

Where p is an embedding parameter and v_0 is an initial approximation of (6) which satisfies the boundary conditions. Obviously, from (6) we have

$$H(V\,0) = L(V) - L(v_0) = 0, \quad (7)$$

$$H(V\,1) = A(V) - f(r) = 0. \quad (8)$$

The changing process of p from zero to unity is just that of $V(r, p)$ from $v_0(r)$ to $v(r)$. In topology, this is called homotopy. We can first use the embedding parameter as a small parameter and assume that the solution of (6) can be written as a power series in

$$V = {}_0 + pV_1 + p^2V_2 + \ldots. \quad (9)$$

Setting p =1, the approximate solution (5) will be concluded as

$$v = \lim_{p\to 1} = V_0 + V_1 + V_2 + \cdots. \quad (10)$$

The combination of the perturbation and homotopy method is called the homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques [11].

Combination of the Approaches: In this section we introduce the process of combination homotopy perturbation and parameterization methods. First we consider $\{e_k(t)\}$ as a basis, which is dense in the space of $C([0,1])$. The continuous control function $c(t)$ can be approximated by a finite combination from elements of this basis [12]. We construct a convex combination as follows:

$$S(x,p) = X^{(n)}(t) - y(t) - p\left(\int_0^t k_1(s,t,x(s),\ldots,x^{(n)}(s),u(s))ds + \int_0^T k_2(s,t,x(s),\ldots,x^{(n)}(s),u(s))ds\right) = 0, \quad p\in[0,1]. \quad (11)$$

Our scheme in the kth iteration considers the control parametrized form as

$$u(t) = \sum_{j=0}^{k} c_j e_j(t) \quad (12)$$

and a power series such as (9), in which $X(t,c_0,c_1,\ldots,c_k)$, i = 0,1,2,..., are unknown functions that must be determined. With substituting power series X and (12) in (11) and equating the terms with identical power of p, we have.

$$\begin{aligned} &p^0: X_0^{(n)}(t) = y(t),\\ &p^1: X_1^{(n)}(t,c_0,c_1,\cdots c_k) = p\Big(\int_0^t k_1(s,t,X_0(s),\ldots,X_0^{(n)}(t),u(s))\,ds + \int_0^1 k_2(s,t,X_0(s),\ldots,X_0^{(n)}(t),u(s))ds,\\ &p^2: X_2^{(n)}(t,c_0,c_1,\cdots c_k) = p\Big(\int_0^t k_1(s,t,X_1(s),\ldots,X_1^{(n)}(t),u(s))\,ds + \int_0^1 k_2(s,t,X_1(s),\ldots,X_1^{(n)}(t),u(s))\,ds,\\ &\vdots \end{aligned} \quad (13)$$

Note that, to obtain $X_i(t,c_0,c_1,\ldots,c_k)$, i = 0,1,2,..., the initial condition (3) is required. The approximate solution of (13), which are depends on the parameters c_j, j = 0,1,...,k can be obtained setting p = 1, as follows

$$x(t,c_0,c_1,\cdots,c_k) = \lim_{p\to 1} X = X_0 + X_1 + X_2 + \cdots \quad (14)$$

Substituting trajectory and control (14) and (12) in (1), the solution of a constraint optimization problem as follows

$$\min_{(c_0,c_1,\cdots c_k)} J_k(c_0,c_1,\cdots c_k) = \int_0^T f_0(t,x(t,c_0,c_1,\cdots,c_k),\sum_{j=0}^{k} c_j e_j(t))dt,$$

$$s.t. \qquad (15)$$

$$x^{(r)}(1,c_0,c_1,\cdots,c_k) = x^{(r)}{}_T,\ 0 \le r \le n-1,$$

may give rise to approximate optimal trajectory and control functions of problem (1)-(3). It is natural to wonder when iterations in the given procedure can be determined. Assuming J_k^* as optimal value of (15) in the k th iteration, a stopping criteria may be considered as follows:

$$\left\| J_{k+1}^* - J_k^* \right\| < \varepsilon, \qquad (16)$$

for a prescribed small positive number ε that should be chosen according to the desired accuracy or if the number of iterations exceeds a predetermined number.

Note that, there is only one difficulty in the process of applying approach which is to obtain the unknown functions in (13) may be difficult, specially for large n and we consider it in the process of the approach.

The above results has been summarized in an algorithm. This algorithm is presented in two stages, initialization step and main steps.

Initialization Step: Choose e>0 for the accuracy desired and a dense basis $\{e_k(t)\}$, for the space $C([0,1])$ and parameter p is in interval [0,1]. Set $m = k = 1$ and go to the main steps.

Main Steps: Step 1. Set $u(s)$ by (12) and go to Step 2.
Step 2. Compute $X_m(t,c_0,c_1,...,c_k)$ by (13) if it is possible and go to Step 3, otherwise, $k = k + 1$ and go Step 1.
Step 3. Compute $(t,c_0,c_1,...,c_k)$ = arg min j_k in (15) by (14) and go to Step 4.
Step 4. If the stopping criteria (16) holds, stop; Otherwise, $m = m + 1$ and go to Step 2.

Numerical Results: In this section, some numerical examples are given to show the efficiency of the proposed algorithm. In all examples, monomial functions $\{t^k\}$ have been considered as dense basis of $C([0,1])$.

Example 1: Consider the following optimal control problem which is minimization of the functional

$$J(x,u) = \int_0^1 (x(t) - t^2 - 1)^2 + (u(t) - t^2)dt,$$

governed by integro-differential equation

$$\dot{x}(t) = 2t + \int_0^t s^2\dot{x}(s)u(s)ds - \int_0^1 s^2 t^6 \dot{x}(s)\, u(s)ds$$

The boundary conditions are

$$x(0) = 1,\ x(1) = 2.$$

The exact optimal trajectory and control functions are $x(t) = t^2 + 1$ and $u(t) = t^2$, respectively. The computed results of applying the proposed algorithm in the previous section have been shown in Table 1. Also the obtained approximate optimal control and trajectory which has been compared to the exact ones can be seen in Fig.1.

Example 2: Consider the following optimal control problem, in which k_1 =0,

$$\text{minimize } J(x,u) = \int_0^1 (x(t) - u(t))^2\, dt,$$

Subject to

$$\dot{x}(t) = e^t - \frac{1}{3}t + \int_0^1 s^2 t\, e^{-2s}\dot{x}(s)u(s)ds,$$

with boundary conditions

$$x(0) = 1,\ x(1) = e.$$

Here $x(t) = 1$, $u(t) = e^t$ are the exact optimal trajectory and control functions, respectively. The results of applying the given algorithm are presented in Table 2. Also, one can observe the approximate optimal trajectory and control functions which is obtained in some iterations of the given algorithm and compared to the exact solutions in Fig.2.

Example 3: In this example the following optimal control problem is considered, in which k_2 = 0,

$$\text{minimize } J(x,u) = \int_0^1 (t\,x(t) - u(t))^2\, dt,$$

Subject to

$$\dot{x}(t) = 1 - \frac{7}{12}t^4 + \int_0^t (s^2 t + s u(s))\,\dot{x}(s)\, ds.$$

Table 1: Numerical results in Example 1.

k	n	J_k
1	3	0.0262
2	3	1.1130×10^{-7}
3	3	2.2843×10^{-8}

Table 2: Numerical results in Example 2.

k	n	J_k
1	2	0.0045
2	2	4.2147×10^{-4}
3	2	4.0135×10^{-4}

Table 3: Numerical results in Example 3.

k	n	J_k
1	2	0.0062
2	2	1.1876×10^{-4}
3	2	1.1091×10^{-4}

Table 4: Numerical results in Example 4.

k	n	J_k
1	3	0.0091
2	3	1.6000×10^{-6}
3	3	1.5950×10^{-6}

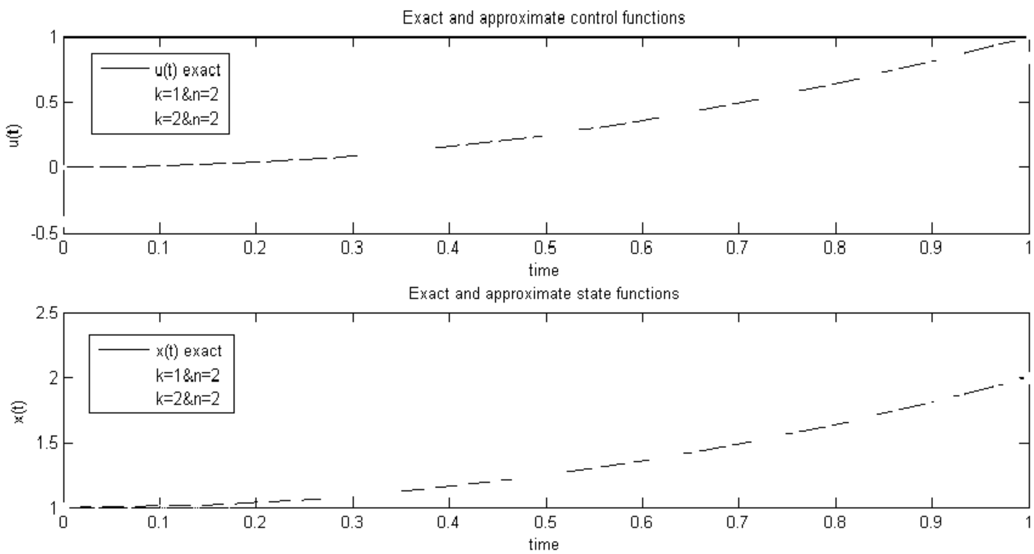

Fig. 1: Exact and approximate optimal controls and trajectories in Example 1.

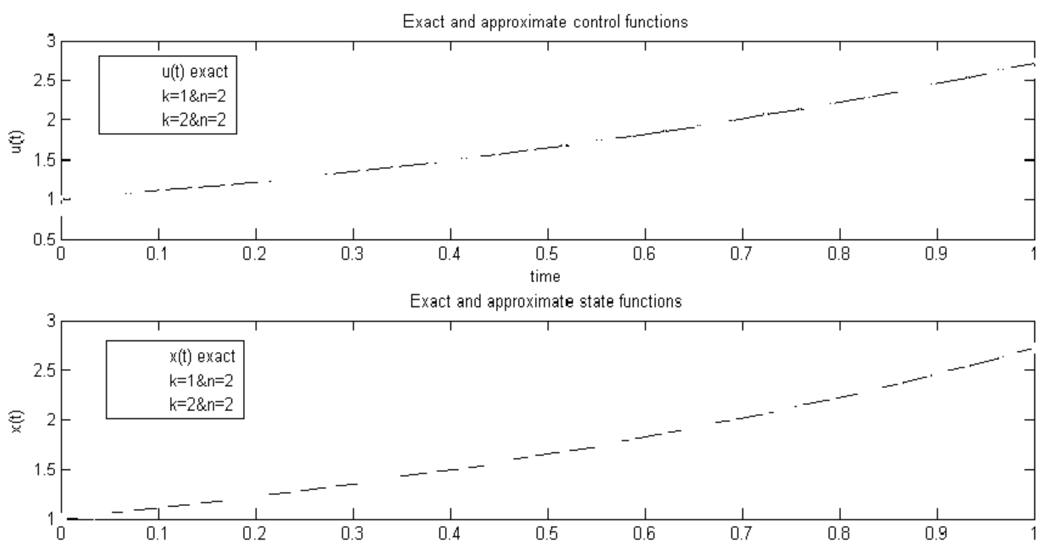

Fig. 2: Exact and approximate optimal controls and trajectories in Example 2.

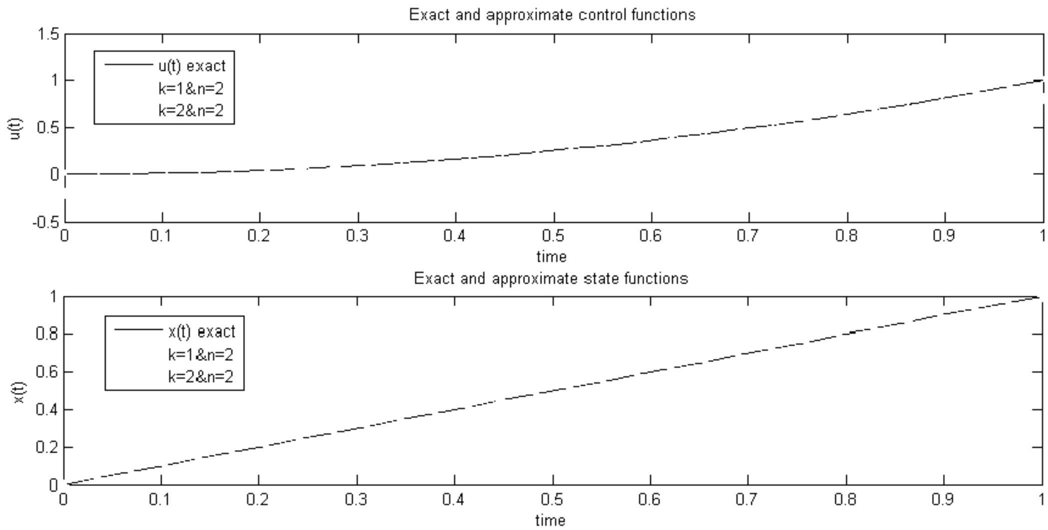

Fig. 3: Exact and approximate optimal controls and trajectories in Example 3.

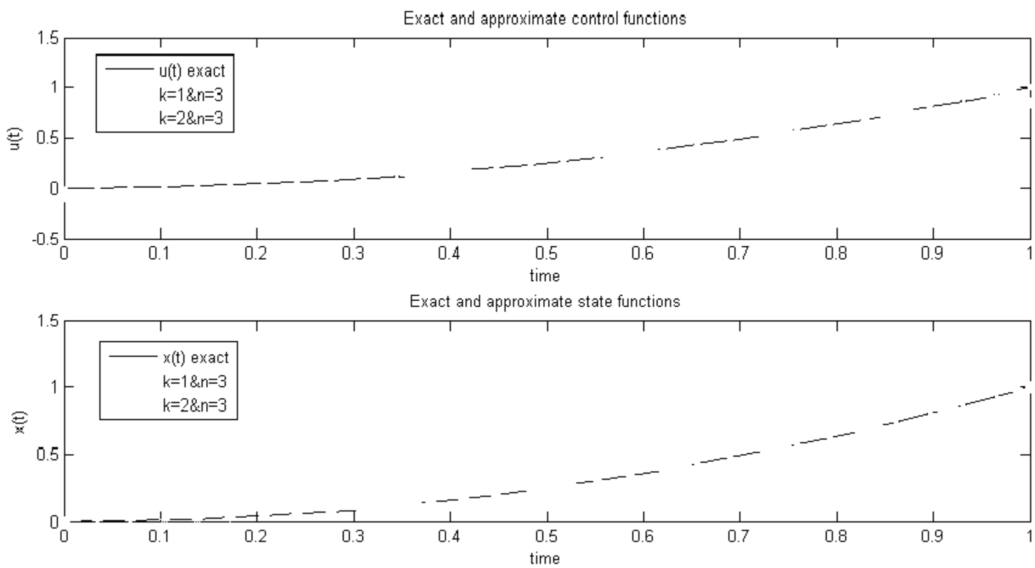

Fig. 4: Exact and approximate optimal controls and trajectories in Example 4.

The boundary conditions are

$$x(0) = 0, x(1) = 1.$$

The exact optimal trajectory and control functions are $x(t) = t$ and $u(t) = t^2$, respectively. The results of applying the algorithm have been shown in Table 3. The comparison of the exact and approximate optimal control and trajectory may be seen in Fig.3.

Example 4: Consider the following optimal control problem which is minimization of the functional, in which $k_1 = 0$,

$$J(x,u) = \int_0^1 (x(t) - t^2)^2 + (u(t) - t^2)^2 \, dt,$$

Subject to

$$\ddot{x}(t) = 2 - \frac{1}{3}t + \int_0^1 s^3 t \, \ddot{x}(s) u(s) \, ds.$$

The boundary conditions are

$$x(0) = 0, \ \dot{x}(0) = 0.$$

$$x(1) = 1, \ \dot{x}(1) = 2.$$

The exact optimal trajectory and control functions are $x(t) = u(t) = t^2$.

The computed results of applying the proposed algorithm in the previous section have been shown in Table 4. Also, one can observe the approximate optimal trajectory and control functions which is obtained in some iterations of the given algorithm and compared to the exact solutions in Fig.4.

CONCLUSION

In this article, the homotopy perturbation method as a powerful tool in solving integro-differential equations is applied for presenting a novel and successful hybrid iterative scheme to find approximate solutions of optimal control problems governed by integro-differential equations. The proposed procedure is simple and effective and the obtained results show that the approximate solutions are near to the exact solutions.

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