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Differential Transform Method for Solving Helmholtz Equation

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Abstract: In this article differential transform method (DTM) is considered to solve Helmholtz equation. This method is a powerful tool for solving a large number of problems. Several illustrative examples are given to demonstrate the effectiveness of the present method. MSC (2010): 65Mxx, 65M99.

Key words: Differential transform method · Helmholtz equation

INTRODUCTION

Consider Helmholtz equation on a given region R in the XY-plane in the following form.

$$\nabla^2 u + f(x, y)u = g(x, y). \qquad (1)$$

Where:
$u(x, y)$ is known on the boundary of R The boundary and initial condition could be given by the following functions.

$$u(0, y) = \psi_1(y), \quad u_x(0, y) = \psi_2(y), \qquad (2)$$

$$u(x, 0) = \psi_3(y), \quad u_y(x, 0) = \psi_4(y). \qquad (3)$$

Where:
$\psi_1(y)$, $\psi_2(y)$, $\psi_3(y)$ and $\psi_4(y)$ are known functions [1].

In this paper, we implement differential transform method [2] for finding the exact solution of the Helmholtz equations. This equation appears in diverse phenomena such as elastic waves in solids including vibrating string, bars, membranes, sound or acoustic, electromagnetic waves and nuclear reactors [3, 4].

Basic Idea of Differential Transform Method: The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [5, 6] as follows. The differential transform of a typical function $u(x, y)$ is the form

$$U(k, h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h} u(x, y)}{\partial x^k \partial y^h}\right]_{(x_0, y_0)}, \qquad (4)$$

where:
$u(x, y)$ is the original function and $U(k, h)$ is the transformed function.

The differential inverse transform of $U(k, h)$ is defined as

$$u(x, y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k, h)(x - x_0)^k (y - y_0)^h, \qquad (5)$$

in a real application and when (x_0, y_0) are taken as (0, 0), then the function $u(x, y)$ is expressed by a finite series and Eq. (5) can be written as

$$u(x, y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!\,h!}\left[\frac{\partial^{k+h} u(x, y)}{\partial x^k \partial y^h}\right] x^k y^h. \qquad (6)$$

Eq. (6) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. In this study, we use the lower case letters to represent the original functions and upper case letters to stand for the transformed functions (T-functions).

From the definitions of Eqs. (4) and (5), it is readily proved that the transformed functions comply with the following basic mathematical operations.

The following theorems that can be easily deduced:

1. If $u(x, y) = g(x, y) \pm h(x, y)$ then $U(k, h) = g(k, h) \pm H(k, h)$

2. If $u(x, y)\ \lambda\ g(x, y)$, then $U(k, h) = \lambda G(k, h)$, where, is λ a constant.

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3. If $u(x,y)=\dfrac{\partial g(x,y)}{\partial x}$, then $U(k,h)=(k+1)\,G(k+1,h)$.

4. If $u(x,y)=\dfrac{\partial g(x,y)}{\partial y}$, then $U(k,h)=(h+1)\,G(k,h+1)$.

5. If $u(x,y)=\dfrac{\partial^{r+s} g(x,y)}{\partial x^r \partial y^s}$, then

$U(k,h)=(k+1)(k+2)\ldots(k+r)$
$\times(h+1)(h+2)\ldots(h+s)\,G(k+r,h+s)$.

6. If $u(x,y)=x^m y^n$, then
$U(k,h)=\delta(k-m)\,\delta(h-n)=\delta(k-m,h-n)$,

where $\delta(k-m,h-n)=\begin{cases}1, & k=m, h=n,\\ o & otherwise.\end{cases}$

7. If $u(x,y)=g(x,y)\,h(x,y)$, then

$$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}G(r,h-s)H(k-r,s).$$

Example 1. Consider the following initial value equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+xu=2+x^3+xy. \quad (7)$$

With initial condition,

$$\begin{cases}u(0,y)=y,\\ u_x(0,y)=0.\end{cases} \quad (8)$$

Taking the differential transform of (7), then

$$\begin{aligned}&(k+1)(k+2)U(k+2,h)\\ &+(h+1)(h+2)U(k,h+2)\\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-1)\delta(h-s)U(k-r,s)\\ &=2\delta(k)\delta(h)+\delta(k-3)\delta(h)\\ &+\delta(k-1)\delta(h-1),\end{aligned} \quad (9)$$

From the initial condition given by Eqs (8).

$$U(0,h)=\begin{cases}1 & h=1,\\ 0 & h=0,2,3,\ldots\end{cases} \quad (10)$$
$$U(1,h)=0,\ \ h=0,1,\ldots.$$

Substituting Eqs. (10) in Eq. (9), all spectra can be found as

$$U(k,h)=\begin{cases}1 & k=2, h=0,\\ 1 & k=0, h=1,\\ 0 & otherwise.\end{cases}$$

We obtained the closed form series solution as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h = y+x^2.$$

Which is the exact solution.

Example 2: Consider the Helmholtz equation as follows [7].

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-u=0, \quad (11)$$

with indicated initial conditions,

$$\begin{cases}u(0,y)=y,\\ u_x(0,y)=y+\cosh y.\end{cases} \quad (12)$$

With the exact solution

$$u(x,y)=x\cosh y+ye^x.$$

Taking the differential transform of (11), then

$$\begin{aligned}&(k+1)(k+2)U(k+2,h)\\ &+(h+1)(h+2)U(k,h+2)-U(k,h)=0.\end{aligned}$$

From the initial condition given by Eq. (12)

$$U(0,h)=\begin{cases}1 & h=1,\\ 0 & h=0,2,\ldots\end{cases} \quad (13)$$

$$U(1,h)=\begin{cases}\dfrac{1}{h!} & h \text{ is even},\\ 1 & h=1,\\ 0 & otherwise.\end{cases}$$

Substituting (13) in (12), all spectra can be found as

$$U(k,h)=\begin{cases}\dfrac{1}{h!} & k=1 \text{ and } h \text{ is even},\\ \dfrac{1}{k!} & h=1 \text{ and } k=0,1,\ldots,\\ 0 & \text{otherwise}.\end{cases}$$

Therefore, the closed form of the solution can be easily written as

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k y^h$$
$$=x\sum_{h=0,2,\ldots}^{\infty}\frac{1}{h!}y^h+y\sum_{k=0,1,\ldots}^{\infty}\frac{1}{k!}x^k=x\cosh y+ye^x.$$

Which is the exact solution.

CONCLUSIONS

In this study, we have introduced differential transform method (DTM) to solve Helmholtz equation. It may be concluded that the method is very powerful and efficient in finding the analytical solutions for a wide class of initial boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems.

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