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Solutions of Singular Boundary Value Problems of Emden-Fowler Type by the Variational Iteration Method

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Abstract: In this paper, approximate-exact solutions of a class of linear and nonlinear Emden-Fowler type singular boundary value problems are presented by the variational iteration method (VIM). The VIM yields solutions in the forms of convergent series with easily computable components. Numerical results explicitly reveal reliability, efficiency and accuracy of the proposed algorithm.

Key words: Boundary value problems · Variational iteration method · Emden-Fowler equation
AMS Classification: 65L05 · 65M99

INTRODUCTION

The Emden-Fowler equations [1-4] are of utmost importance in nonlinear sciences and are frequently used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Moreover, such equations are widely applicable in mathematical physics, astrophysics, theory of stellar structure, thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents. Several techniques including decomposition, finite difference, homotopy analysis, power series, Lie group analysis and Ritz have been employed to tackle such problems, see [1-9] and the references therein. Most of the techniques used so far are coupled with their inbuilt deficiencies like calculation of the s0-called Adomian's polynomials, lengthy calculations, limited convergence, inaccurate and divergent results at specific points. He [7] developed the variational iteration method (VIM) which has been applied [1-21] to a wide class of initial and boundary value problems. The basic motivation of this paper is the extension of this very reliable technique for solving the Lane–Emden equation. It is worth mentioning that Hosseini and Nasabzadeh [10] introduced a modification of Adomian's decomposition method for solving Lane-Emden singular problems.

The Lane-Emden equations is obtained from Emden-Fowler equation which is of the form

$$y''+\frac{r}{x}y'+F(x,y)=g(x), \qquad (1)$$

subject to the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \qquad (2)$$

(where $F(x, y)$ is a continuous function and $g(x)$ is a given function), by taking $r = 2$, $F(x, y) = y^n$.

Variational Iteration Method (VIM): The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained.

This approximation converges rapidly to an accurate solution. To illustrate the basic concepts of the VIM, we consider the following nonlinear differential equation:

$$Ly + Ny = g(x), \qquad (3)$$

Where:
L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to the VIM, we can construct a correction functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_a^x \lambda\{\mathrm{L}y_n(\tau) + N\tilde{y}_n(\tau) - g(\tau)\}d\tau, \qquad n \geq 0, \qquad (4)$$

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Where:
λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the subscript n denotes the *nth-order* approximation, $\tilde{y}_n$ is considered as a restricted variation, i.e. $\delta\tilde{y}_n$.

Analysis of the Method: To solve Eq. (1) by means of He's variational iteration method, we construct a correction functional,

$$y_{n+1}(x) = y_n(x) + \int_a^x \lambda(\tau)\left\{y_n''(\tau) + \frac{r}{\tau}y_n'(\tau) + \tilde{F}(\tau, y_n(\tau)) - \tilde{g}(\tau)\right\}d\tau. \quad (5)$$

To determine the optimal value of $\lambda(\tau)$, we continue as follows;

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_a^x \lambda(\tau)\left\{y_n''(\tau) + \frac{r}{\tau}y_n'(\tau) + \tilde{F}(\tau, y_n(\tau)) - \tilde{g}(\tau)\right\}d\tau, \quad (6)$$

or

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_a^x \lambda(\tau)\left\{y_n''(\tau) + \frac{r}{\tau}y_n'(\tau)\right\}d\tau, \quad (7)$$

which gives

$$\delta y_{n+1}(x) = \left[1 - \lambda'(x) + \frac{r}{\tau}\lambda(x)\right]\delta y_n(x) + \delta\lambda(x)y_n'(x) + \int_a^x \delta y_n\left[\lambda''(\tau) - r\frac{\tau\lambda'(\tau) - \lambda(\tau)}{\tau^2}\right]d\tau = 0, \quad (8)$$

Where:
$\tilde{y}_n$ is considered as restricted variations, which mean $\delta\tilde{y}_n$ 0. Its stationary conditions can be obtained as follows

$$1 - \lambda'(\tau) + \frac{r}{x}\lambda(x) = 0, \qquad \lambda(x) = 0, \qquad \lambda''(\tau)r\frac{\tau\lambda'(\tau) - \lambda(\tau)}{\tau^2}|_{\tau=x} = 0, \quad (9)$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda(x) = \frac{\tau^r}{(r-1)x^{r-1}} - \frac{\tau}{(\tau-1)}, \quad (10)$$

and the following variational iteration formula is obtained

$$y_{n+1}(x) = y_n(x) + \int_a^x \frac{\tau^r}{(r-1)x^{r-1}}\frac{\tau}{(\tau-1)}\left\{y_n''(\tau) + \frac{r}{\tau}y_n'(\tau) + \tilde{F}(\tau, y_n(\tau)) - \tilde{g}(\tau)\right\}d\tau, \quad (11)$$

Numerical Illustrations: In this section, we demonstrate the effectiveness of the proposed method with five illustrative examples. In all examples, $s_n(x)$, denotes the n^{th} partial sum

$$\text{sn}(x) = \sum_{i=0}^{n} y_i.$$

Example 1: Let us first consider the linear singular boundary value problem:

$$y'' + \frac{2}{x}y' = 3\frac{2}{x'} \qquad y(0) = 0, \quad y(1) = -\frac{1}{2}.$$

The exact solution is:

$$y(x) = \frac{x^2}{2} - x. \quad (12)$$

Solution via the method in [2]:
We choose

$$L^{-1} = \int_0^x\int_0^t(.)\,dsdt,$$

and recursive relation

$$y_0 = ax + L^{-1}\left(3 - \frac{2}{x}\right),$$
$$yn = -L^{-1}\left(\frac{2}{x}y'_{n-1}\right) \qquad n \geq 1,$$

Where:
$\alpha = y'$ is unknown.
Now by applying the method, we have:

$$y_0 = ax + \int_0^x\int_0^t\left(3 - \frac{2}{s}\right)dsdt = ax = \int_0^x (3s - 2\ln(s))|_0^t\,dt$$

where the above integral is divergent.

Solution via the method in [22]:
We choose

$$L^{-1} = \int_0^x\int_1^t(.)\,dsdt,$$

and recursive relation

$$y_0 = q(x)\left(-\frac{1}{2}\right) - q(x)L^{-1}\left(3 - \frac{2}{x}\right)|_{x=1} + L^{-1}\left(3 - \frac{2}{x}\right),$$
$$y_n = -q(x)L^{-1}\left(\frac{2}{x}y'_{n-1}\right)|_{x=1} + L^{-1}\left(\frac{2}{x}y'_{n-1}\right) \qquad n \geq 1,$$

Where: $q(x) = x$.

Table 1:

n	$\|\|s_n(x) - y(x)\|\|$
2	1.02
4	4.1
8	50
16	3000

Solution via the method in [23]:

Table 2: Shows the obtained results for n=7

x_i	$\|s_7(x_i) - y(x_i)\|$
1.1	0.722330560
1.2	1.466223422
1.3	2.09545938
1.4	2.63497650
1.5	3.10232406
1.6	3.5004298
1.7	3.8483370
1.8	4.0512897
1.9	3.6418928

The obtained results are shown in Table 1.
Suppose that:

$$\lambda(\tau) = (\tau - x),$$
$$y_0(x) = A + Bx,$$

we have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x (\tau - x)\left\{ y_n''(\tau) + \frac{2}{\tau} y_n'(\tau) - 3 + \frac{2}{\tau} \right\} d\tau,$$

By the above iteration formula we have the following approximate solution:

$$y_1(x) = A + Bx + \int_0^x (\tau - x)\left\{ \frac{2B}{\tau} - 3 + \frac{2}{\tau} \right\} d\tau = (2B+2)\tau - 3\frac{\tau^2}{2} + 3x\tau - (2B+2)x\ln(\tau)\Big|_0^x$$

Where:
the above integral is divergent.

Solution via the Proposed Method: If the proposed method is used for solving this problem then according to (10) and (11), we obtain:

$$\lambda(\tau) = \left(\frac{\tau^2}{x} - \tau \right),$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\tau^2}{x} - \tau \right)\left\{ y_n''(\tau) + \frac{2}{\tau} y_n'(\tau) - 3 + \frac{2}{\tau} \right\} d\tau,$$

We start by initial approximation $y_0(x) = A + Bx$. By the above iteration formula we have,

$$y_1(x) = A + Bx + \int_0^x \left(\frac{\tau^2}{x} - \tau \right)\left\{ \frac{2B}{\tau} - 3 + \frac{2}{\tau} \right\} d\tau = A + Bx - Bx + \frac{x^2}{2} - x = A + \frac{x^2}{2} - x$$

By imposing the boundary conditions at x = 0, we find A = 0, i.e., the exact solution (12) is obtained by only one iteration.

Example 2: We consider the linear singular Emden-Fowler equation:

$$y'' + \frac{2}{x} y' = 110x^8, \qquad y(1) = 1, \quad y(2) = 2^{10}.$$

The exact solution is:

$$y(x) = x^{10}. \tag{13}$$

Solution via the method in [2]:
We choose

$$L^{-1} = \int_1^x \int_1^t (.)\, ds dt,$$

and recursive relation

$$y_0 = 1 + ax + L^{-1}\left(110x^8\right),$$

Where:
$a = y'(1)$ is unknown.
We choose

$$L^{-1} = \int_1^x \int_2^t (.)\, ds dt,$$

and recursive relation

$$y_0 = 1 + q(x)\left(2^{10} - 1\right) - q(x)L^{-1}\left(110x^8\right)\Big|_{x=2} + L^{-1}\left(110x^8\right),$$
$$y_n = -q(x)L^{-1}\left(\frac{2}{x} y'_{n-1}\right)\Big|_{x=1} + L^{-1}\left(\frac{2}{x} y'_{n-1}\right) \qquad n \geq 1,$$

Where: $q(x) = x - 1$.
The results are shown in Table 3, for n =7.
Suppose that:

$$\lambda(\tau) = (\tau - x),$$
$$y_0(x) = A + Bx,$$

We have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_1^x (\tau - x)\left\{ y_n''(\tau) + \frac{2}{\tau} y_n'(\tau) - 110\tau^8 \right\} d\tau,$$

The results are shown in Table 4, for n = 7.

Table 3:

x_i	$\lvert s_7(x_i) - y(x_i)\rvert$
1.1	0.000289
1.2	0.00110
1.3	0.00188
1.4	0.00229
1.5	0.00227
1.6	0.00183
1.7	0.00122
1.8	0.00066
1.9	0.00022

Table 4:

x_i	$\lvert s_7(x_i) - y(x_i)\rvert$
1.1	0.006742
1.2	0.012361
1.3	0.017110
1.4	0.021144
1.5	0.024480
1.6	0.026888
1.7	0.027703
1.8	0.025515
1.9	0.017735

Solution via the proposed method:

If the proposed method is used for solving this problem then according to (10) and (11), we obtain:

$$\lambda(\tau)\left(\frac{\tau^2}{x}-\tau\right),$$

We begin with an arbitrary initial approximation $y_0(x) = A + Bx$. By the above iteration formula we have,

$$y_1(x) = A + Bx + \int_1^x \left(\frac{\tau^2}{x}-\tau\right)\left\{\frac{2^B}{\tau}-110\tau^8\right\}d\tau = A + Bx + \frac{10-B}{x}$$
$$+(2B-11) - Bx + x10 = (A+2B-11) + \frac{10-B}{x} + x^{10}$$

Applying the boundary conditions at $x = 1$ and $x = 2$ yields $A = 9$ and $B = 10$. In fact, we obtain the exact solution (13) with only one iteration.

Example 3: We consider the nonlinear singular Emden-Fowler equation:

$$y'' + \frac{2}{x}y' - 6y^2 = 6 + \frac{2}{x} - 6\left(x^2+x\right)^2, \qquad y(0)=1, \quad y(1)=2.$$

The exact solution is:

Suppose that:

$$\lambda(\tau) = (\tau - x),$$
$$y_0(x) = A + Bx,$$

We have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x (\tau - x)\left\{y_n''(\tau) + \frac{2}{\tau}y_n'(\tau)' - 6yn^2 - 6 - \frac{2}{\tau} + 6\left(\tau^2-\tau\right)^2\right\}d\tau,$$

By the above iteration formula we have the following approximate solution:

$$y_1(x) = A + Bx + \int_0^x (\tau - x)\left\{\frac{2B}{\tau} - 6(A+Bx)^2 - 6 - \frac{2}{\tau} + 6\left(\tau^2-\tau\right)^2\right\}d\tau,$$

Where:

the above integral is divergent.

Solution via the Proposed Method: If the proposed method is used for solving this problem then according to (10) and (11), we obtain:

$$\lambda(\tau)\left(\frac{\tau^2}{x}-\tau\right),$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\tau^2}{x} - x\right)\left\{\begin{matrix} y_n''(\tau) + \frac{2}{\tau}y_n'(\tau)' \\ -6y_{n^2} - 6 - \frac{2}{\tau} + 6\left(\tau^2-\tau\right)^2 \end{matrix}\right\}d\tau,$$

We start by initial approximation $y_0(x) = A + Bx$. The obtained results are shown in Table 5.

Example 4: Consider the linear singular boundary value problem:

$$y'' + \frac{5}{x}y = 6\frac{5}{x}, \qquad y(0)=0, \quad y(1) = -\frac{1}{2}.$$

$$y_{n+1}(x) = y_n(x) + \int_1^x \left(\frac{\tau^2}{x}-\tau\right)\left\{y_n''(\tau) + \frac{2}{\tau}y_n'(\tau) - 110\tau^8\right\}d\tau,$$

The exact solution is:

$$y(x) = \frac{x^2}{2} - x. \tag{14}$$

Solution via the method in [2]:

We choose

$$L^{-1}\int_0^x\int_0^t (.)\,ds\,dt,$$

and recursive relation

Table 5:

x_i	$\lvert y_1(x_i) - y(x_i)\rvert$	$\lvert y_2(x_i) - y(x_i)\rvert$	$\lvert y_3(x_i) - y(x_i)\rvert$	$\lvert y_4(x_i) - y(x_i)\rvert$
1.1	8.62×10^{-9}	8.19×10^{-13}	4.92×10^{-17}	2.03×10^{-21}
1.2	1.07×10^{-6}	8.70×10^{-10}	4.44×10^{-13}	1.56×10^{-16}
1.3	1.76×10^{-5}	5.08×10^{-8}	9.25×10^{-11}	1.16×10^{-13}
1.4	1.22×10^{-4}	8.87×10^{-7}	4.04×10^{-9}	1.26×10^{-11}
1.5	5.28×10^{-4}	7.85×10^{-6}	7.34×10^{-8}	4.70×10^{-10}
1.6	1.63×10^{-3}	4.40×10^{-5}	7.46×10^{-7}	8.65×10^{-9}
1.7	3.86×10^{-3}	1.74×10^{-4}	4.89×10^{-6}	9.42×10^{-8}
1.8	7.01×10^{-3}	4.92×10^{-4}	2.16×10^{-5}	6.49×10^{-7}
1.9	8.50×10^{-3}	8.86×10^{-4}	5.78×10^{-5}	2.58×10^{-6}

Table 6:

n	$\lVert s_n(x) - y(x)\rVert$
2	20.1
4	400
8	150000
16	8×10^{-9}

Solution via the method in [23]:

$$y_0 = ax + L^{-1}\left(6 - \frac{5}{x}\right),$$

$$y_n = -L^{-1}\left(\frac{5}{x} y'_{n-1}\right) \qquad n \geq 1,$$

Where:

$a = y'(0)$ is unknown.

Now by applying the method, we have:

$$y_0 = ax + \int_0^x \int_0^t \left(6 - \frac{5}{s}\right) ds dt = ax + \int_0^x \left(6s - 5\ln(s)\right)\Big|_0^t dt$$

Where:

the above integral is divergent.

Solution via the method in [22]:

We choose

$$L^{-1}\int_0^x \int_1^t (.)\, ds dt,$$

and recursive relation

$$y_0 = q(x)\left(-\frac{1}{2}\right) - q(x) L^{-1}\left(6 - \frac{5}{x}\right)\Big|_{x=1} + L^{-1}\left(6 - \frac{5}{x}\right),$$

$$y_n = -q(x) L^{-1}\left(\frac{5}{x} y'_{n-1}\right)\Big|_{x=1} + L^{-1}\left(\frac{5}{x} y'_{n-1}\right) \qquad n \geq 1,$$

Where: $q(x) = x$.

The results are shown in Table 6.

Suppose that:

$$\lambda(\tau) = (\tau - x),$$
$$y_0(x) = A + Bx,$$

We have the following iteration formula:

$$y_{n+1}(x) = y_n(x) + \int_0^x (\tau - x)\left\{y_n''(\tau) + \frac{2}{\tau} y_n'(\tau) - 6 + \frac{5}{\tau}\right\} d\tau.$$

By the above iteration formula we have the following approximate solution:

$$y_1(x) = A + Bx + \int_0^x (\tau - x)\left\{\frac{2B}{\tau} - 6 + \frac{5}{\tau}\right\} d\tau = (2B+5)\tau - 3\tau^2 + 6x\tau - (2B+5)x\ln(\tau)\Big|_0^x$$

Where:

the above integral is divergent.

Solution via the proposed method:

If the proposed method is used for solving this problem then according to (10) and (11), we obtain:

$$\lambda(\tau)\left(\frac{\tau^5}{4x^4} - \frac{\tau}{4}\right),$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\frac{\tau^5}{4x^4} - \frac{\tau}{4}\right)\left\{y_n''(\tau) + \frac{2}{\tau} y_n'(\tau) - 6 + \frac{5}{\tau}\right\} d\tau,$$

We begin with an arbitrary initial approximation $y_0(x) = A + Bx$.. So, we obtain,

$$y_1(x) = A + Bx + \int_0^x \left(\frac{\tau^2}{x} - \tau\right)\left\{\frac{2B}{\tau} - 6 + \frac{5}{\tau}\right\} d\tau = A + Bx - Bx + \frac{x^2}{2} - x = A + \frac{x^2}{2} - x$$

By imposing the boundary conditions at $x = 0$ yields $A = 0$, i.e., the exact solution (14) is found by only one iteration.

Table 7:

x_i	$\lvert s_7(x_i) - y(x_i)\rvert$ in [24]	$\lvert s_7(x_i) - y(x_i)\rvert$ in [2]	$\lvert s_7(x_i) - y(x_i)\rvert$ in [22]	$\lvert y_7(x_i) - y(x_i)\rvert$ in [23]	$\lvert y_7(x_i) - y(x_i)\rvert$ by proposed method
1.1	4.23×10^{-4}	6.40×10^{-9}	8.00×10^{-10}	6.40×10^{-9}	1.45×10^{-17}
1.2	6.63×10^{-4}	1.33×10^{-8}	7.00×10^{-10}	1.34×10^{-8}	3.00×10^{-17}
1.3	6.41×10^{-4}	2.08×10^{-8}	3.81×10^{-9}	2.08×10^{-8}	4.69×10^{-17}
1.4	4.26×10^{-4}	2.86×10^{-8}	6.82×10^{-9}	2.86×10^{-8}	6.46×10^{-17}
1.5	1.44×10^{-4}	3.65×10^{-8}	8.74×10^{-9}	3.65×10^{-8}	8.40×10^{-17}
1.6	9.72×10^{-5}	4.34×10^{-8}	8.89×10^{-9}	4.35×10^{-8}	1.03×10^{-16}
1.7	2.34×10^{-4}	4.76×10^{-8}	7.50×10^{-9}	4.77×10^{-8}	1.22×10^{-16}
1.8	2.49×10^{-4}	4.58×10^{-8}	4.99×10^{-9}	4.58×10^{-8}	1.33×10^{-16}
1.9	1.59×10^{-4}	3.25×10^{-8}	2.32×10^{-9}	3.25×10^{-8}	1.14×10^{-16}

Example 5: Consider the following linear problem [22].

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0,$$

with the boundary condition

$$y(1) = 1, \quad y(2) = 1.$$

It is easy to see that the exact solution is $y(x) = = x - \frac{x\ln(x)}{2\ln(2)}$.

Table 7. shows the absolute error at each test point between the exact solution and the seventh partial sum S_7 [2, 22, 24] and the seventh approximate solution y_7. This shows that the proposed method is much more efficient than the other mentioned methods in [2, 22-24].

CONCLUSION

In this paper, variational iteration method (VIM) has been successfully employed to obtain the approximate solution of boundary Emden-Fowler equations with singular behavior. The method was used in a direct way without using linearization, perturbation or restrictive assumptions. In linear cases, the exact solution has been obtained by only one iteration. It may be concluded that the VIM is very powerful and efficient for finding analytical as well as numerical solutions for a wide class of linear and nonlinear boundary singular differential equations. The VIM provides more realistic series solutions that converge very rapidly in real physical problems.

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