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Variational Iteration Method for Nonlinear Whitham-Broer-Kaup Equations Using Adomian's Polynomials

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Abstract: In this paper, we apply modified variational iteration method (VIMAP) to find travelling wave solutions of Whitham-Broer-Kaup (WBK) equations. The proposed modification is made by introducing Adomian's polynomials in the correction functional of the VIM. The use of Lagrange multiplier coupled with Adomian's polynomials is the clear advantage of this technique over the traditional decomposition method. Numerical results explicitly reveal the reliability of proposed VIMAP.

Key words: Variational iteration method · Adomian's polynomials · Whitham-Broer-Kaup (WBK) equation · Nonlinear problems · Traveling wave solutions
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INTRODUCTION

The rapid growth of nonlinear sciences [1-26] witnesses a reasnable number of new and modified versions of some traditional algorithms. He [9-13] developed the variational iteration (VIM) method which is highly suitable for the problems arising in nonlinear sciences. G. Adomian [1] proposed decomposition method which was appropriatly modied by Wazwaz [24-26]. In these methods the solution is given in an infinite series usually converging to an accurate solution, see [2-13, 15-17, 19-23] and the references therein. The basic motivation of this work is to apply the variational iteration method (VIM) coupled with Adomian's polynomials (VIMAP) to find travelling wave solutions of Whitham-Broer-Kaup (WBK) equations [22] which arise quite frequently in mathematical physics, nonlinear sciences and is of the form

$$u_t + uu_x + v_x + \beta u_{xx} = 0 \quad (1)$$

$$v_t + (uv)_x + au_{xxx} - \beta v_{xx} = 0,$$

Where the field of horizontal velocity is represented by $u= u(x,t)$, $v = v(x,t)$ is the height that deviate from equilibrium position of liquid and a,β are constants which represent different diffusion power. This idea has been used first by Abbasbandy [2, 3] to solve quadratic Riccati differential equation and Klein-Gordon equation and subsequently by Noor and Mohyud-Din [16, 17, 19] for finding solutions of a large number of singular and nonsingular initial and boundary value problems. In this method the correction functional is developed [1, 2, 16, 17, 19] and the Lagrange multipliers are calculated optimally via variational theory. The Adomian's polynomials for the nonlinear terms are introduced in the correction functional and can be calculated according to the specific algorithms set in [24-26]. It is shown that the proposed VIMAP provides the solution in a rapid convergent series with easily computable components. Numerical results explicitly reveal the complete reliability of the proposed VIMAP.

Variational Iteration Method (VIM): To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \quad (1)$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method [2-13, 15-17, 19-23], we can construct a correct functional as follows

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$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(L u_n(s) + N \tilde{u}_n(s) - g(s))ds, \quad (2)$$

Where λ is a Lagrange multiplier [2-13, 15-17, 19-23], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [2-13, 15-17, 19-23]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

Adomian's Decomposition Method (ADM)

Consider the differential equation [24-26]

$$Lu + Ru + Nu = g \quad (3)$$

Where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L, Nu represents the nonlinear terms and g is the source term. Applying the inverse operator L^{-1} to both sides of (3) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu),$$

Where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [33-35] defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

Where the components $u_n(x)$ are usually determined recurrently by using the relation

$$u_0 = f$$
$$u_{k+1} = L^{-1}(Ru_k) - L^{-1}(Nu_k), \ k \geq 0.$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n,$$

Where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in [33-35] which yields

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right) F\left(\sum_{i=0}^{n} \left(\lambda^i u_i\right)\right)_{\lambda=0}, \quad n = 0,1,2,\cdots.$$

For further details about the Adomian's decomposition method, see [33-35] and the references therein.

Variational Iteration Method Using Adomian's Polynomials (VIMAP): To illustrate the basic concept of the proposed VIMAP, we consider the following general differential equation (4)

$$Lu + Nu = g(x), \quad (4)$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to variational iteration method [1-3, 5-11, 16-30], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(L u_n(s) + N \tilde{u}_n(s) - g(s))ds, \quad (5)$$

Where λ is a Lagrange multiplier [5-11] , which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (5) is called as a correct functional. We define the solution $u(x)$ by the series

$$u(x) = \sum_{i=0}^{\infty} u_i(x),$$

and the nonlinear term

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_i),$$

Where A_n are the so-called Adomian's polynomials and can be generated for all type of nonlinearities according to the algorithm developed in [33-35] which yields the following

$$A_n = \left(\frac{1}{n!}\right)\left(\frac{d^n}{d\lambda^n}\right)F(u(\lambda))|_{\lambda=0},$$

Hence, we obtain

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda \,(\, L u_n(x) + \sum_{n=0}^{\infty} A_n - g(x)\,)\, dx, \tag{6}$$

Which is the variational iteration method using Adomian's polynomials (VIMAP) and is formulated by the elegant coupling of variational iteration method and the so-called Adomian's polynomials.

Solution Procedure: Consider Whitham-Broer-Kaup (WBK) equation (1) with initial conditions

$$u(x,0) = \lambda - 2Bk\coth(k\xi), \quad v(x,0) = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi), \tag{5}$$

Where $B = \sqrt{\alpha + \beta^2}$ $\xi = x + x_0$ and x_0, k,λ are arbitrary constants. Applying variational iteration method (VIM) on (1, 5). The correction functional is given by

$$\begin{cases} u_{n+1}(x,t) = \lambda - 2Bk\coth(k\xi) + \displaystyle\int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} + \tilde{u}_n \frac{\partial \tilde{u}_n}{\partial x} + \frac{\partial \tilde{v}_n}{\partial x} + \beta \frac{\partial^2 \tilde{u}_n}{\partial x^2}\right) ds, \\ v_{n+1}(x,t) = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi) + \displaystyle\int_0^t \lambda(s)\left(\frac{\partial v_n}{\partial s} + (\tilde{u}_n \tilde{v}_n)_x + \alpha \frac{\partial^3 \tilde{u}_n}{\partial x^3} - \beta \frac{\partial^2 \tilde{v}_n}{\partial x^2}\right) ds. \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers are identifird as $\lambda(s) = -1$ consequently

$$\begin{cases} u_{n+1}(x,t) = \lambda - 2Bk\coth(k\xi) - \displaystyle\int_0^t \left(\frac{\partial u_n}{\partial s} + u_n \frac{\partial u_n}{\partial x} + \frac{\partial v_n}{\partial x} + \beta \frac{\partial^2 u_n}{\partial x^2}\right) ds, \\ v_{n+1}(x,t) = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi) - \displaystyle\int_0^t \left(\frac{\partial v_n}{\partial s} + (u_n v_n)_x + \alpha \frac{\partial^3 u_n}{\partial x^3} - \beta \frac{\partial^2 v_n}{\partial x^2}\right) ds. \end{cases}$$

Applying variational itertaion method using Adomian's polynomials (VIMAP), we get

$$\begin{cases} u_{n+1}(x,t) = \lambda - 2Bk\coth(k\xi) - \displaystyle\int_0^t \left(\frac{\partial u_n}{\partial s} + \sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} \frac{\partial v_n}{\partial x} + \beta \sum_{n=0}^{\infty} \frac{\partial^2 u_n}{\partial x^2}\right) ds, \\ v_{n+1}(x,t) = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi) - \displaystyle\int_0^t \left(\frac{\partial v_n}{\partial s} + \sum_{n=0}^{\infty} B_n + \alpha \sum_{n=0}^{\infty} \frac{\partial^3 u_n}{\partial x^3} - \beta \sum_{n=0}^{\infty} \frac{\partial^2 v_n}{\partial x^2}\right) ds, \end{cases}$$

Where A_n and B_n are the Adomian's polynomials which can be evaluated by using specific algorithm developed in []. Consequently, following approximants are obtained

$$\begin{cases} u_0(x,t) = \lambda - 2Bk\coth(k\xi), \\ v_0 = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi), \end{cases}$$

$$\begin{cases} u_1(x,t) = \lambda - 2Bk\coth(k\xi) - 2Bk^2\lambda t \operatorname{csc}h^2(k(x+x_0)), \\ v_1 = -2B(B+\beta)k^2 \operatorname{csc}h^2(k\xi) - 2B(B+\beta)k^2 t(-\lambda + 2Bk\coth(k\xi))\operatorname{csc}h^2(k\xi) \\ \quad - 4B(\alpha + B\beta + \beta^2)k^4 t(2 + \cosh(2k(x+x_0)))\operatorname{csc}h^4(k\xi), \end{cases}$$

$$\begin{cases} u_2 = \dfrac{Bk^3t^2}{2}\operatorname{csc}h^5(k\xi)((-44\alpha k^2 - 44B\beta k^2 - 44\beta^2k^2 - B\lambda - \beta\lambda + \lambda^2)\cosh(k\xi) \\ -(4\alpha k^2 + 4B\beta k^2 + 4\beta^2k^2 - B\lambda - \beta\lambda + \lambda^2)\cosh(3k\xi) - (6B^2k + 6B\beta k - 6Bk\lambda - 6\beta k\lambda)\sinh(k\xi) \\ -(2B^2k + 2B\beta k - 2Bk\lambda - 2\beta k\lambda)\sinh(3k\xi), \\ v_2 = \dfrac{Bk^2t^2}{8}\operatorname{csc}h^6(k\xi)(4B^3k^2 + 4B^2\beta k^2 - 528\alpha\beta k^4 - 528B\beta^2k^4 - 528\beta^3k^4 - 12\alpha k^2\lambda + 8B^2k^2\lambda \\ -16B\beta k^2\lambda - 24\beta^2k^2\lambda - 3B\lambda^2 - 3\beta\lambda^2 - (416\alpha\beta k^4 + 416B\beta^2k^4 + 416\beta^3k^4 - 8\alpha k^2\lambda + 8B^2k^2\lambda \\ -8B\beta k^2\lambda - 16\beta^2k^2\lambda - 4B\lambda^2 - 4\beta\lambda^2)\cosh(2k\xi) - (4B^3k^2 + 4B^2\beta k^2 + 16\alpha\beta k^4 + 16B\beta^2k^4 + 16\beta^3k^4 \\ -4\alpha k^2\lambda\cosh(4k\xi) + 8B\beta k^2\lambda - 8\beta^2k^2\lambda + B\lambda^2 + \beta\lambda^2)\cosh(4k\xi) - (32\alpha Bk^3 + 112B^2\beta k^3 + 112B\beta^2k^3 \\ +8B^2k\lambda + 8B\beta k\lambda + 80\alpha k^3\lambda)\sinh(2k\xi) - (8\alpha Bk^3 + 16B^2\beta k^3 + 16B\beta^2k^3 \\ -4B^2k\lambda - 4B\beta k\lambda + 8\alpha k^3\lambda)\sinh(4k\xi)), \\ \vdots \end{cases}$$

Hence, the closed form solutions are given as

$$u(x,t) = \lambda - 2Bk\coth(k(\xi - \lambda t)), \tag{6}$$

$$v(x,t) = -2B(B+\beta)k^2 \operatorname{csc}h^2(k(\xi - \lambda t)), \tag{7}$$

Where $B = \sqrt{\alpha + \beta^2}$ and $\xi = x + x_0$ and x_0, k, λ are arbitrary constants. As a special case, if $a = 1$ and $\beta = 0$, WBK equations can be reduced to the modified Boussinesq (MB) equations.We shall second consider the initial conditions of the MB equations

$$u(x,0) = \lambda - 2k\coth(k\xi), \quad v(x,0) = -2k^2 \operatorname{csc}h^2(k\xi), \tag{8}$$

Where $\xi = x + x_0$ being arbitrary constant. Procedding as before, we obtain exact solution as follows

Table 5.1: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the exact solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $\beta = 0$ and $x_0 = 10$, for the approximate solution of the WBK equation

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	1.04892E-04	4.25408E-04	9.71992E-04	1.75596E-03	2.79519E-03
0.3	9.64474E-05	3.91098E-04	8.93309E-04	1.61430E-03	2.56714E-03
0.5	8.88312E-05	3.60161E-04	8.22452E-04	1.48578E-03	2.36184E-03
$\lvert v - \varphi_n \rvert$					
0.1	6.41419E-03	1.33181E-03	2.07641E-02	2.88100E-02	3.75193E-02
0.3	5.99783E-03	1.24441E-02	1.93852E-02	2.68724E-02	3.49617E-02
0.5	5.61507E-03	1.16416E-02	1.81209E-02	2.50985E-02	3.26239E-02

Table 5.2: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the analytical solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 10$, for the approximate solution of the MB equation

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	8.16297E-07	3.26243E-06	7.33445E-06	1.30286E-05	2.03415E-05
0.3	7.64245E-07	3.05458E-06	6.86758E-06	1.22000E-05	1.90489E-05
0.5	7.16083E-07	2.86226E-06	6.43557E-06	1.14333E-05	1.78528E-05
$\lvert v - \varphi_n \rvert$					
0.1	5.88676E-05	1.18213E-04	1.78041E-04	2.38356E-04	2.99162E-04
0.3	5.56914E-05	1.11833E-04	1.68429E-04	2.25483E-04	2.83001E-04
0.5	5.27169E-05	1.05858E-04	1.59428E-04	2.13430E-04	2.67868E-04

Table 5.3: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the analytical solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 10$, for the approximate solution of the ALW equation

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u - \phi_n \rvert$					
0.1	8.02989E-06	3.23228E-05	7.32051E-05	1.31032E-04	2.06186E-04
0.3	7.38281E-06	2.97172E-05	6.73006E-05	1.20455E-04	1.89528E-04
0.5	6.79923E-06	2.73673E-05	6.19760E-05	1.10919E-04	1.74510E-04
$\lvert v - \varphi_n \rvert$					
0.1	4.81902E-04	9.76644E-04	1.48482E-03	2.00705E-03	2.54396E-03
0.3	4.50818E-04	9.13502E-04	1.38858E-03	1.87661E-03	2.37815E-03
0.5	4.22221E-04	8.55426E-04	1.30009E-03	1.75670E-03	2.22578E-03

$$u(x,t) = \lambda - 2k\coth(k\xi-\lambda t), \quad v(x,t) = -2k^2 \operatorname{csch}^2(k\xi-\lambda t),$$

Where k,λ are constants to be determined and x_0 is an arbitrary constant. In the last example, if $a = 0$ and $\beta = 1/2$, WBK equations can be reduced to the approximate long wave (ALW) equation in shallow water. We can compute the ALW equation with the initial conditions

$$u(x,0) = \lambda - k\coth(k\xi), \quad v(x,0) = -k^2 \operatorname{csch}^2 (k\xi),$$

Where k is constant to be determined and $\xi = x + x_0$ Procedding as before, we obtain exact solution as follows

$$u(x,t) = \lambda - k\coth(k\xi - \lambda t), \quad v(x,t) = -2k^2 \operatorname{csc} h^2 (k\xi - \lambda t)$$

Where k,λ are constants to be determined and $\xi = x + x_0$, x_0 is an arbitrary constant. In order to verify numerically whether the proposed methodology lead to higher accuracy, we evaluate the numerical solutions using the n-term approximation. Tables 5.1-5.3 show the difference of analytical solution and numerical solution of the absolute error. We achieved a very good approximation with the actual solution of the equations by using 5 terms only of the proposed MVIM.

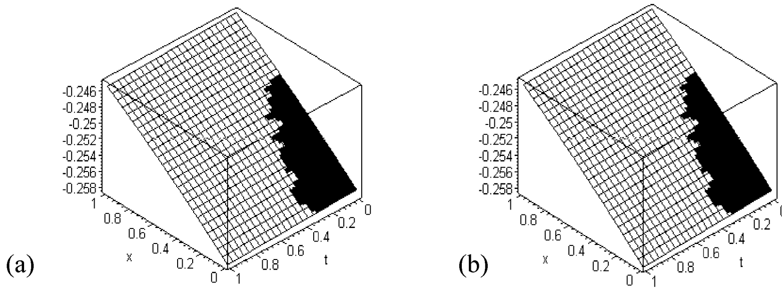

Fig. 1: The surface shows the solution $u(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $\beta = 0$ and $x_0 = 10$ (a) exact solution (b) approximate solution

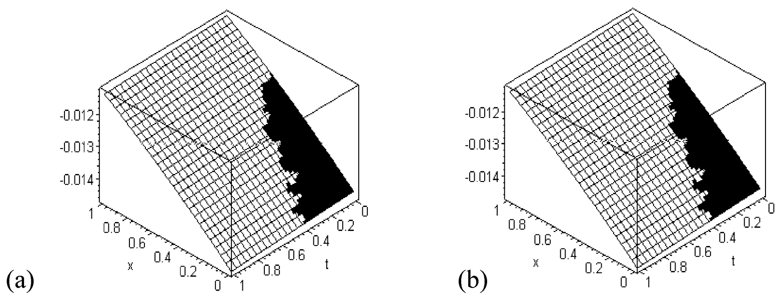

Fig. 2: The surface shows the solution $v(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $\beta = 0$ and $x_0 = 10$ (a) exact solution (b) approximate solution

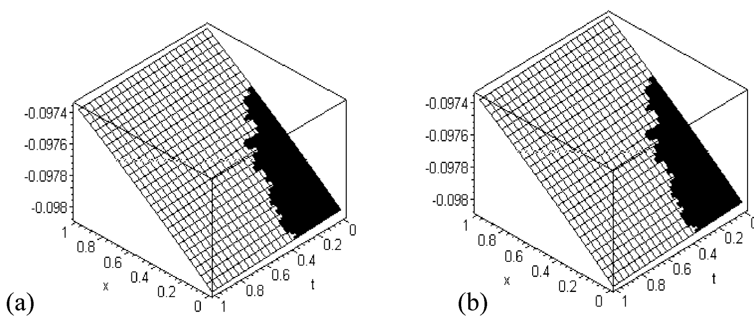

Fig. 3: The surface shows the solution $u(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 20$ (a) exact solution (b) approximate solution

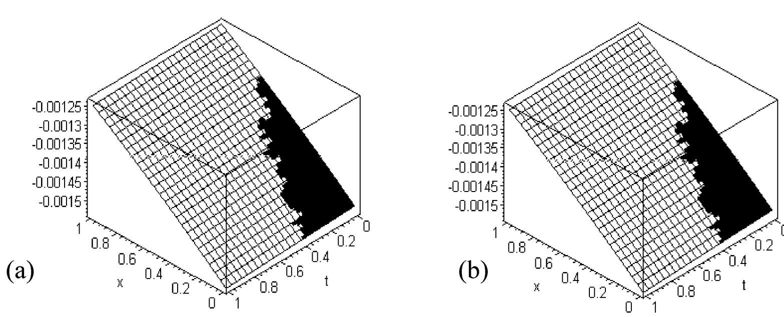

Fig. 4: The surface shows the solution $v(x, t)$ when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 20$ (a) exact solution (b) approximate solution

CONCLUSION

In this paper, we applied variational iteration method using Adomian's polynomials (VIMAP) to find travelling wave solutions of Whitham-Broer-Kaup (WBK) equation. The proposed modification is made by introducing Adomian's polynomials in the correction functional of the VIM. Numerical results explicitly reveal the complet reliability of VIMAP.

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