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Over Determined Systems of Complex Partial Differential Equations in Theoretical Physics

[1]Nasir Taghizadeh and [2]Ahmad Neirameh

[1]Department of Mathematics Faculty of science Guilan University, Rasht, Iran
[2]Islamic Azad University Gonbad Kavoos Branch, Gonbad, Iran

Abstract: In this manuscript we want to offer the new generalization of over determined systems of complex partial differential equations. It is proved that many global properties of analytic functions are true for solutions of the Vekua system in special cases.

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INTRODUCTION

In this paper we consider the system

$$\frac{\partial W}{\partial \bar{z}_{js}} = A_{js}(z_{1s}, z_{2s}, \cdots, z_{ns})\varphi(W) \quad s, j = 1,2,\cdots,n \tag{1}$$

Extension of some properties of analytic functions on several complex variables during the past century is the main goal for many researchers. Such as the I.N.Vekua [6] has studied the system.

$$\frac{\partial W}{\partial \bar{z}} = A(z)W + B(z)\bar{W} \tag{2}$$

And extended many properties of functions analytic in $z \in C$ to generalized solutions of (2), under suitable hypotheses of coefficients $A(z)$ and $B(z)$. Therefore the generalized solutions of (2) are called generalized analytic functions (see [1-5]). And some properties of the system (1) are considered, as the necessary and sufficient condition for the extended of a non vanished solution.

Generalized System: In this section we suppose that the following conditions hold:

$$(\alpha)\ A_{js}(z_{1s}, z_{2s}, \ldots, z_{ns}) \in c^{\infty}(G);\ s,j = 1,2,..,n$$

(b) $\varphi(W)$ is analytic in W
(c) $G \subset C^{n}$ Is domain of holomorphy

Theorem: Assume that $\varphi(W) \neq 0$ and that the system (1) has a evolution. Then the coefficients $A_{js}(z_s)$ and $z_s = (z_{1s}, z_{2s}, \ldots, z_{ns})$ satisfy in following condition.

$$\frac{\partial A_{js}}{\partial \bar{z}_{ks}} = \frac{\partial A_{ks}}{\partial \bar{z}_{js}}, \quad s, j, k = 1,2,\cdots,n \tag{3}$$

Proof: Let $W(z_{1s}, z_{2s}, \ldots, z_{ns}) = W(z_s) \in c^2(G)$ be a solution of (1). Then

$$\frac{\partial^2 W}{\partial \bar{z}_{js}\,\partial \bar{z}_{ks}} = \frac{\partial A_{js}}{\partial \bar{z}_{ks}}\varphi(W) + A_{js}A_{ks}\varphi'(W)\varphi(W) \tag{4}$$

And

$$\frac{\partial^2 W}{\partial \bar{z}_{ks}\,\partial \bar{z}_{js}} = \frac{\partial A_{ks}}{\partial \bar{z}_{js}}\varphi(W) + A_{ks}A_{js}\varphi'(W)\varphi(W) \tag{5}$$

Since $\varphi(W) \neq 0$, by comparing (4) and (5) we have (3). Suppose that $\varphi(W)$ is defined in a simply connected domain of the complex plane C and $\varphi(W) \neq 0$. Take an indefinite integral $F(W)$ of $\frac{1}{\varphi(W)}$ and set

$$F(W) = c.e^{F(W)},\ c = const. \in C.$$

By definition, it is

$$f' = \frac{f}{\varphi} \tag{6}$$

Corresponding Author: Ahmad Neirameh, Islamic Azad University, Gonbad Kavoos Branch, Iran.

Theorem: Let $\Omega\subset G$ be an open set and K be a compact subset of Ω, such that $\Omega\backslash K$ connected. If $W(z_s)$ is a c^2-solution of (1) in $\Omega\backslash K$, then the function

$$W_f = f\,|W(z_s)|$$

Can be extended continuously in the whole of $\Omega\backslash K$.

Proof: First we consider the system of the following form

$$\frac{\partial W}{\partial z_{js}} = A_{js}(z_{1s}, z_{2s}, \cdots, z_{ns}) = A_{js}(z_s) \quad s, j = 1,2,\cdots,n \tag{7}$$

and note that (as in the section 2) it is solvable in $c^\infty(G)$. Take a fixed solution $W_0(z_s)\in c^\infty(G)$ of this system and set:

$$\frac{\partial \Phi}{\partial z_{js}} = |\, f'(W)\frac{\partial W}{\partial z_{js}} - f(W)A_{js}\,|\, e^{-\omega_0} = (\frac{f}{\varphi}.A_{js}\varphi - f.A_{js})e^{-\omega_0} \tag{8}$$

Where $W_0(z_s)$ is the given c^2-solution of (1) in $\Omega\backslash K$ Then $\Phi(z_s)$. is defined in $\Omega\backslash K$. From (6) it follows

$$\tilde{W}_f(z_s) := \tilde{\phi}(z_s)e^{\omega_0}$$

For $\in \Omega\backslash K$.

This means that $\Phi(z_s)$ is analytic in $\Omega\backslash K$. Because of the Hartogs extension theorem $\Phi(z_s)$ has an analytic extension $\Phi(z_s)$ in Ω. Then

$$f\,|W(z_s)| = \Phi(z_s)e^{\omega_0(z_s)}$$

is the required extension of $\tilde{W}_f(z_s)$ in Ω. Q.e.d.

Now suppose that there exists f^{-1} (or at least there exists a single valued branch f^{-1} o), then from theorem (2.2) it follows

Theorem (Hartogs Extension Theorem): Every C^2-solution of (1) in $\Omega\backslash K$ can be extended continuously in the whole of Ω as a solution of this system.

Proof: Define the function $\Phi(z_s)$ by formula (7), where $W_0(z_s)$ is the given solution in $\Omega\backslash K$, then $\Phi(z_s)$ is analytic in $\Omega\backslash K$. From the definition we have:

$$f\,|W(z_s)| = \Phi(z_s)e^{\omega_0(z_s)} \tag{9}$$

Because of the existence of f^{-1}, we have

$$W(z_s) = f^{-1}\Phi(z_s)e^{\omega_0(z_s)} \tag{10}$$

Now Define:

$$\tilde{W}(z_s) = f^{-1}\tilde{\Phi}(z_s)e^{\omega_0(z_s)}\,| \tag{11}$$

Where $\Phi(z_s)$ is the analytic extension of $\Phi(z_s)$ in Ω. Then $\tilde{W}(z_s)$ is defined for all$\in \Omega$. Furthermore we have

$$\frac{\partial \tilde{W}}{\partial z_{js}} = (f^{-1})\tilde{\Phi}(z_s)e^{\omega_0(z_s)}\frac{\partial \omega_0}{\partial z_{js}} = \frac{\tilde{\Phi}.A_{js}.e^{\omega_0}}{f'} = \frac{\varphi}{f}A_j\tilde{\phi}e^{\omega_0} \tag{12}$$

On the other hand, by definition (11) we have

$$f\,|\tilde{W}(z_s)| = \tilde{\Phi}(z_s)e^{\omega_0(z_s)} \tag{13}$$

Hence (by virtue of (12)) it follows

$$\frac{\partial \tilde{W}}{\partial z_{js}} = A_{js}.\varphi.\frac{f\left|\tilde{W}\right|}{f\left|\tilde{W}\right|} = A_{js}.\varphi, \;\; j,s = 1,2,\cdots,n \tag{14}$$

Thus, $\tilde{W}(z_s)$ is a solution of (1) in Ω. From the definition it follows

$$\tilde{W}(z_s) = W(z_s)$$

For $z_s\in\Omega\backslash K$. The proof is complete.

Theorem: Suppose that

- F as in Proposition 3.2.
- $\Gamma(D) = \partial D_1 \times \partial D_2 \times \cdots \times \partial D_n$ as in Theorem 2.1.
- $W(z_s) \in c^1(\bar{D})$ is a solution of (1).

Then we have

$$W(z_s) = f^{-1}\{\int_{\Gamma(D)} F(\xi, z_s) f\,|W(\xi)|\, d\xi_1 \cdots d\xi_n, \;\; s = 1,2,\cdots,n\} \tag{15}$$

for $z_s\in D$,where $F(\xi, z_s)$ and $\Gamma(D)$ are defined as follows

$$F(\xi, z_s) = \frac{1}{(2\pi i)^n}\frac{e^{w_0(z_s) - w_0(\xi)}}{(\xi_1 - z_{1s})\cdots(\xi_n - z_{ns})}$$

And

$$\Gamma(D) = \partial D_1 \times \partial D_2 \times \cdots \times \partial D_n$$

Proof: Applying the Cauchy integral formula for the analytic functions, then from (10) it follows (15).Q.e.d.

As an example to applicate the results obtained in section 2 we consider the system

$$\frac{\partial W}{\partial \bar{z}_{js}} = A_{js}(z_{1s}, z_{2s}, \cdots, z_{ns})W^m \qquad s, j = 1,2,\cdots,n \tag{16}$$

In this case we have $\varphi(W) = w^m$.

In the sequel we consider only c^2- and non vanished solutions of (16) and denote by (Ω) the set of such solutions in $\Omega \subseteq G$. Then $\varphi(W) \neq 0$. Applying the theorem (2.1) we obtain:

Proposition: For the existence of a solution $W(z_s) \in v^*(G)$ it is necessary that the following condition

$$\frac{\partial A_{js}}{\partial \bar{z}_{ks}} = \frac{\partial A_{ks}}{\partial \bar{z}_{js}}, \quad s, j, k = 1,2,\cdots,n$$

Holds in G.Now set

$$F(W) = \frac{W^{1-m}}{1-m}$$

Then $F(W)$ is an infinite integral of

$$\frac{1}{\varphi(W)} = W^{-m}$$

Hence the function $f(W)$ defined by (3.4) becomes

$$f(W) = c.e^{\frac{W^{1-m}}{1-m}}$$

Without loss generality we may choose c=1 and have

$$f(W) = e^{\frac{W^{1-m}}{1-m}}$$

Suppose that $W(z_s) \in v^*(G)$.As in (3.6) we define

$$\phi(z_s) = e^{|\frac{W^{1-m}}{1-m} - \omega_0|}$$

Then) ϕ is analytic in Ω. By definition $\phi \neq 0$ in Ω, thus

$$\phi \in \Upsilon(\Omega)$$

Where Υ(Ωis the space of non vanished analytic functions in Ω.

Take a single-valued analytic branch of ϕ (it is possible if Ω is simply connected), we have

$$W^{1-m} = (1-m)\{\ln\phi + \omega_0\}$$

Hence

$$W = (1-m)^{\frac{1}{(1-m)}}\{\ln\phi + \omega_0\}^{\frac{1}{(1-m)}}$$

Thus we have

Proposition: For a given $W(z_s) \in v^*(n)$, there exists a function $\phi \in \Upsilon(\Omega)$ such that $W(z_s)$can be represented by the formula (17).

As a direct corollary of (17) we obtain.

Theorem (Hartogs Extension Theorem): Let $\Omega \subseteq G$ be an open set and K be a compact subset of Ω such that $\Omega \backslash K$ is connected. Then every $W(z_s) \in v^*(\Omega \backslash K)$ can be extended to a solution $\tilde{W}(z_s)$ of the system (16) in Ω.

Proof: Define the function ϕ by formula (7), where $W(z_s) \in v^*(\Omega \backslash K)$. It is easily to show that $\Omega \in \Upsilon^*(\Omega \backslash K)$. Then $\frac{1}{\phi}$ is analytic in

Because of the Hartogs extension theorem, ϕ and $\Omega \backslash K$ can be extended analyticaly in the whole of Ω. Denote by ϕ the analytic extension of ϕ, then $\bar{\phi}$ is the analytic extension ofϕ. Hence it follows that

$$\bar{\phi}(z_s) \neq 0$$

For $z^s \in \Omega$ or $\qquad \bar{\phi} \in \Upsilon^*(\Omega)$

Taking a single-valued analytic branch of $\bar{\phi}$ and define

$$\tilde{W} = (1-m)^{\frac{1}{1-m}}\{Ln\tilde{\Phi} + \omega_0\}^{\frac{1}{1-m}}$$

Then

$$\frac{\partial \tilde{W}}{\partial z_{js}} = (1-m)^{\frac{1}{1-m}} \cdot \frac{1}{1-m}\{Ln\tilde{\Phi} + \omega_0\}^m \cdot \frac{\partial \omega_0}{\partial z_{js}}$$

$$= A_{js}(1-m)^{\frac{m}{1-m}}\{Ln\tilde{\Phi} + \omega_0\}^{\frac{m}{1-m}}$$

$$= A_{js}.\tilde{W}^m$$

Hence $\tilde{W}$ is a solution of (16) in Ω. By definition it is $\tilde{W} = W$ in Ω\K.

Thus $\tilde{W}$ is the required extension of W.Q.e.d.

CONCLUSIONS

The generalizations of these systems have many potential applications in partial differential equations.

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