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Modified Variational Iteration Method for Wbk Equations

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Abstract In this paper, we apply modified variational iteration method (MVIM) to find travelling wave solutions of Whitham-Broer-Kaup (WBK) equation. The proposed modification is made by introducing He's polynomials in the correction functional of the VIM. The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method. Numerical results explicitly reveal the reliability of MVIM.

Key words: Modified variational iteration method · He's polynomials · Whitham–Broer–Kaup (WBK) equation · Nonlinear problems · Traveling wave solutions
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INTRODUCTION

In the recent past, He [1-6] developed variational iteration (VIM) and homotopy perturbation (HPM) methods which are highly suitable for the problems arising in nonlinear sciences [1-10, 11-20]. In a later work Ghorbani et al. [10] introduced He's polynomials which are compatible with Adomian's polynomials but are easier to calculate and are more user friendly. It is to be highlighted that He's polynomials are calculated from He's homotopy perturbation method (HPM) [10, 12, 14, 17]. Most recently, Noor and Mohyud-Din [14-17] made the elegant coupling of He's polynomials with the correctional functional of VIM. This modified version (MVIM) [14-17] is quite efficient in solving the nonlinear problems. The basic inspiration of this paper is the extension of modified variational iteration method (MVIM) to find travelling wave solutions of Whitham–Broer–Kaup (WBK) equations [3, 11, 19] which arise quite frequently in mathematical physics and nonlinear sciences and is of the form.

$$\begin{aligned} &u_t + uu_x = v_x + \beta u_{xx} = 0, \\ &v_t = (uv)_x + \alpha u_{xx} - \beta v_{xx} = 0, \end{aligned} \tag{1}$$

Where the field of horizontal velocity is represented by $u = u(x,t)$, $v = v(x,t)$ is the height that deviate from equilibrium position of liquid and α, β are constants which represent different diffusion power. Numerical results explicitly reveal the complete reliability of the proposed MVIM.

Modified Variational Iteration Method (MVIM): To illustrate the basic concept of the MVIM, we consider the following general differential equation.

$$Lu + Nu = g(x) \tag{2}$$

Where L is a linear operator, N a nonlinear operator and g(x) is the forcing term. According to VIM [1-3, 5-9, 11, 14-19], we can construct a correction functional as follows.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\right)d\xi, \tag{3}$$

Where λ is a Lagrange multiplier [1, 3, 5, 6], $\tilde{u}_n$ is a restricted variation; (3) is called a correction functional. Now, we apply He's polynomials [4]

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p\int_0^x \lambda(\xi)\left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n)\right) d\xi - \int_0^x \lambda(\xi)\, g(\xi)\, d\xi, \tag{4}$$

Which is the MVIM [14-17] and is formulated by the coupling of VIM and He's polynomials. The comparison of like powers of p gives solutions of various orders.

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Solution Procedure: Consider Whitham–Broer–Kaup (WBK) equation (1) with initial conditions

$$u(x,0) = \lambda - 2Bk\coth(k\xi),\quad v(x,0) = -2B(B+\beta)k^2\,\mathrm{csch}^2(k\xi), \tag{5}$$

Where $B=\sqrt{\alpha+\beta^2}$ and $\xi = x + x_0$ and x_0, k, λ are arbitrary constants. Applying variational iteration method (VIM) on (1, 5). The correction functional is given by.

$$\begin{cases} u_{n+1}(x,t) = \lambda - 2Bk\coth(k\xi) + \\ \int_0^t \lambda(s)\left(\dfrac{\partial u_n}{\partial s} + \tilde{u}_n\dfrac{\partial \tilde{u}_n}{\partial x} + \dfrac{\partial \tilde{v}_n}{\partial x} + \beta\dfrac{\partial^2 \tilde{u}_n}{\partial x^2}\right)ds, \\ v_{n+1}(x,t) = -2B(B+\beta)k^2\,csch^2(k\xi) + \\ \int_0^t \lambda(s)\left(\dfrac{\partial v_n}{\partial s} + (\tilde{u}_n\tilde{v}_n)_x + \alpha\dfrac{\partial^3 \tilde{u}_n}{\partial x^3} - \beta\dfrac{\partial^2 \tilde{v}_n}{\partial x^2}\right)ds. \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers are identifird as $\lambda(s) = -1$, consequently

$$\begin{cases} u_{n+1}(x,t) = \lambda - 2Bk\coth(k\xi) - \\ \int_0^t \left(\dfrac{\partial u_n}{\partial s} + u_n\dfrac{\partial u_n}{\partial x} + \dfrac{\partial v_n}{\partial x} + \beta\dfrac{\partial^2 u_n}{\partial x^2}\right)ds, \\ v_{n+1}(x,t) = -2B(B+\beta)k^2\,csch^2(k\xi) - \\ \int_0^t \left(\dfrac{\partial v_n}{\partial s} + (u_n v_n)_x + \alpha\dfrac{\partial^3 u_n}{\partial x^3} - \beta\dfrac{\partial^2 v_n}{\partial x^2}\right)ds. \end{cases}$$

Applying modified variational itertaion method (MVIM), we get.

$$\begin{cases} u_0 + pu_1 + \cdots = \lambda - 2Bk\coth(k\xi) - \\ p\int_0^t \left(\left(\dfrac{\partial u_0}{\partial s} + p\dfrac{\partial u_1}{\partial s} + \cdots\right) + (u_0 + pu_1 + \cdots)\left(\dfrac{\partial u_0}{\partial x} + p\dfrac{\partial u_1}{\partial x} + \cdots\right)\right)ds - \\ p\int_0^t \left(\left(\dfrac{\partial v_0}{\partial x} + p\dfrac{\partial v_1}{\partial x} + \cdots\right) + \beta\left(\dfrac{\partial^2 u_0}{\partial x^2} + p\dfrac{\partial^2 u_1}{\partial x^2} + \cdots\right)\right)ds, \\ v_0 + pv_1 + \cdots = -2B(B+\beta)k^2\,csch^2(k\xi) - \\ p\int_0^t \left(\left(\dfrac{\partial v_0}{\partial s} + p\dfrac{\partial v_1}{\partial s} + \cdots\right) + \left((u_0 + pu_1 + \cdots)(v_0 + pv_1 + \cdots)\right)_x\right)ds. - \\ p\int_0^t \left(\alpha\left(\dfrac{\partial^3 u_0}{\partial x^3} + p\dfrac{\partial^3 u_1}{\partial x^3} + \cdots\right) - \beta\left(\dfrac{\partial^2 v_0}{\partial x^2} + p\dfrac{\partial^2 v_1}{\partial x^2} + \cdots\right)\right)ds. \end{cases}$$

Comparing the co-efficient of like powers of p, consequently, following approximants are obtained.

$$p^{(0)}: \begin{cases} u_0(x,t) = \lambda - 2Bk\coth(k\xi), \\ v_0 = -2B(B+\beta)k^2\,csch^2(k\xi), \end{cases}$$

$$p^{(1)}: \begin{cases} u_1(x,t) = \lambda - 2Bk\coth(k\xi) - 2Bk^2\lambda t\,csch^2(k(x+x_0)), \\ v_1 = -2B(B+\beta)k^2\,csch^2(k\xi) - 2B(B+\beta)k^2 t(-\lambda + \\ 2Bk\coth(k\xi))csch^2(k\xi) - 4B(\alpha + B\beta + \beta^2)k^4 t(2 + \\ \cosh(2k(x+x_0)))csch^4(k\xi), \end{cases}$$

$$p^{(2)}: \begin{cases} u_2 = \dfrac{Bk^3t^2}{2}csch^5(k\xi)((-44\alpha k^2 - 44B\beta k^2 - 44\beta^2k^2 - \\ B\lambda - \beta\lambda + \lambda^2)\cosh(k\xi) - (4\alpha k^2 + 4B\beta k^2 + 4\beta^2k^2 - B\lambda - \\ \beta\lambda + \lambda^2)\cosh(3k\xi) - (6B^2k + 6B\beta k - 6Bk\lambda - 6\beta k\lambda)\sinh(k\xi) - \\ (2B^2k + 2B\beta k - 2Bk\lambda - 2\beta k\lambda)\sinh(3k\xi), \\ v_2 = \dfrac{Bk^2t^2}{8}csch^6(k\xi)(4B^3k^2 + 4B^2\beta k^2 - 528\alpha\beta k^4 - 528B\beta^2k^4 - \\ 528\beta^3k^4 - 12\alpha k^2\lambda + 8B^2k^2\lambda - 16B\beta k^2\lambda - 24\beta^2k^2\lambda - 3B\lambda^2 - 3\beta\lambda^2 - \\ (416\alpha\beta k^4 + 416B\beta^2k^4 + 416\beta^3k^4 - 8\alpha k^2\lambda + 8B^2k^2\lambda - 8B\beta k^2\lambda - \\ 16\beta^2k^2\lambda - 4B\lambda^2 - 4\beta\lambda^2)\cosh(2k\xi) - (4B^3k^2 + 4B^2\beta k^2 + 16\alpha\beta k^4 + \\ 16B\beta^2k^4 + 16\beta^3k^4 - 4\alpha k^2\lambda\cosh(4k\xi) + 8B\beta k^2\lambda - 8\beta^2k^2\lambda + B\lambda^2 + \\ \beta\lambda^2)\cosh(4k\xi) - (32\alpha Bk^3 + 112B^2\beta k^3 + 112B\beta^2k^3 + 8B^2k\lambda + \\ 8B\beta k\lambda + 80\alpha k^3\lambda)\sinh(2k\xi) - (8\alpha Bk^3 + 16B^2\beta k^3 + 16B\beta^2k^3 - \\ 4B^2k\lambda - 4B\beta k\lambda + 8\alpha k^3\lambda)\sinh(4k\xi)), \\ \vdots \end{cases}$$

Hence, the closed form solutions are given as

$$u(x,t) = \lambda - 2Bk\coth(k(\xi - \lambda t)), \tag{6}$$

$$v(x,t) = -2B(B+\beta)k^2\,\mathrm{csch}^2(k(\xi - \lambda t)), \tag{7}$$

Where $B=\sqrt{\alpha+\beta^2}$ and $\xi = x + x_0$ and x_0, k, λ are arbitrary constants. As a special case, if $\alpha = 1$ and $\beta = 0$, WBK equations can be reduced to the modified Boussinesq (MB) equations.We shall second consider the initial conditions of the MB equations

Table 3.1: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the exact solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.004$, $\alpha = 1.5$, $\beta = 1.5$ and $x_0 = 10$, for the approximate solution of the WBK equation

t_i/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u-\varphi_n\rvert$					
0.1	1.04892E-04	4.25408E-04	9.71992E-04	1.75596E-03	2.79519E-03
0.3	9.64474E-05	3.91098E-04	8.93309E-04	1.61430E-03	2.56714E-03
0.5	8.88312E-05	3.60161E-04	8.22452E-04	1.48578E-03	2.36184E-03
$\lvert v-\varphi\rvert$					
0.1	6.41419E-03	1.33181E-03	2.07641E-02	2.88100E-02	3.75193E-02
0.3	5.99783E-03	1.24441E-02	1.93852E-02	2.68724E-02	3.49617E-02
0.5	5.61507E-03	1.16416E-02	1.81209E-02	2.50985E-02	3.26239E-02

Table 3.2: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the analytical solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$ $\alpha = 1.$, $\beta = 0$ and $x_0 = 10$, for the approximate solution of the MB equation

t/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u-\varphi_n\rvert$					
0.1	8.16297E-07	3.26243E-06	7.33445E-06	1.30286E-05	2.03415E-05
0.3	7.64245E-07	3.05458E-06	6.86758E-06	1.22000E-05	1.90489E-05
0.5	7.16083E-07	2.86226E-06	6.43557E-06	1.14333E-05	1.78528E-05
$\lvert v-\varphi_n\rvert$					
0.1	5.88676E-05	1.18213E-04	1.78041E-04	2.38356E-04	2.99162E-04
0.3	5.56914E-05	1.11833E-04	1.68429E-04	2.25483E-04	2.83001E-04
0.5	5.27169E-05	1.05858E-04	1.59428E-04	2.13430E-04	2.67868E-04

Table 3.3: The numerical results for $\phi_n(x,t)$ and $\varphi_n(x,t)$ in comparison with the analytical solution for $u(x,t)$ and $v(x,t)$ when $k = 0.1$, $\lambda = 0.005$ $\alpha = 0$, $\beta = 0.5$ and $x_0 = 10$, for the approximate solution of the ALW equation

t/x_i	0.1	0.2	0.3	0.4	0.5
$\lvert u-\varphi_n\rvert$					
0.1	8.02989E-06	3.23228E-05	7.32051E-05	1.31032E-04	2.06186E-04
0.3	7.38281E-06	2.97172E-05	6.73006E-05	1.20455E-04	1.89528E-04
0.5	6.79923E-06	2.73673E-05	6.19760E-05	1.10919E-04	1.74510E-04
$\lvert v-\varphi_n\rvert$					
0.1	4.81902E-04	9.76644E-04	1.48482E-03	2.00705E-03	2.54396E-03
0.3	4.50818E-04	9.13502E-04	1.38858E-03	1.87661E-03	2.37815E-03
0.5	4.22221E-04	8.55426E-04	1.30009E-03	1.75670E-03	2.22578E-03

$$u(x,0) = \lambda - 2k\coth(k\xi), \quad v(x,0) = -2k^2\operatorname{csch}^2(k\xi), \quad (8)$$

Where $\xi = x = x_0$ being arbitrary constant. Procedding as before, we obtain exact solution as follows

$$u(x,0) = \lambda - 2k\coth(k\xi - \lambda t), \quad v(x,0) = -2k^2\operatorname{csch}^2(k\xi - \lambda),$$

Where k,λ are constants to be determined and x_0 is an arbitrary constant. In the last example, if $\alpha = 0$ and $\beta = 1/2$, WBK equations can be reduced to the approximate long wave (ALW) equation in shallow water. We can compute the ALW equation with the initial conditions.

$$u(x,0) = \lambda - k\coth(k\xi), \quad v(x,0) = -k^2\operatorname{csch}^2(k\xi),$$

Where k,λ is constant to be determined and $\xi = x + x_0$. Procedding as before, we obtain exact solution as follows

$$u(x,t) = \lambda - k\coth(k\xi - \lambda t), \quad v(x,t) = -2k^2\operatorname{csch}^2(k\xi - \lambda),$$

Where k,λ are constants to be determined and $\xi = x +_0$, x_0 is an arbitrary constant. In order to verify numerically whether the proposed methodology lead to higher accuracy, we evaluate the numerical solutions using the n-term approximation. Tables 3.1-3.3 show the difference of analytical solution and numerical solution of the absolute error. We achieved a very good approximation with the actual solution of the equations by using 5 terms only of the proposed MVIM.

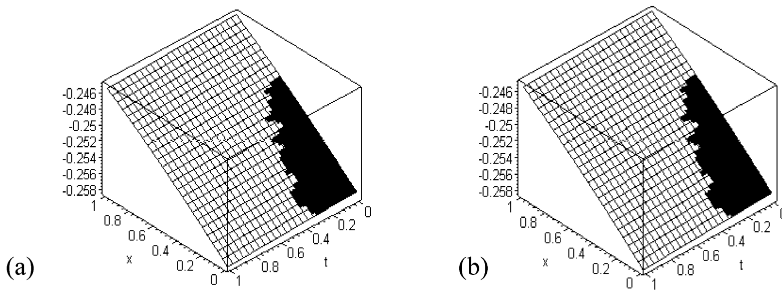

Fig. 1: The surface shows the solution u(x, t) when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $\beta = 0$ and $x_0 = 10$ (a) exact solution (b) approximate solution

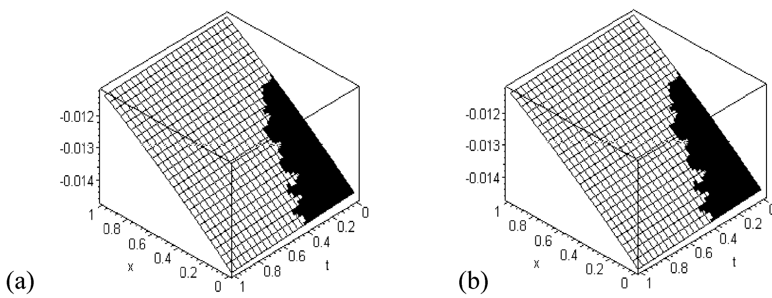

Fig. 2: The surface shows the solution v(x, t) when $k = 0.1$, $\lambda = 0.005$, $a = 1$, $\beta = 0$ and $x_0 = 10$ (a) exact solution (b) approximate solution

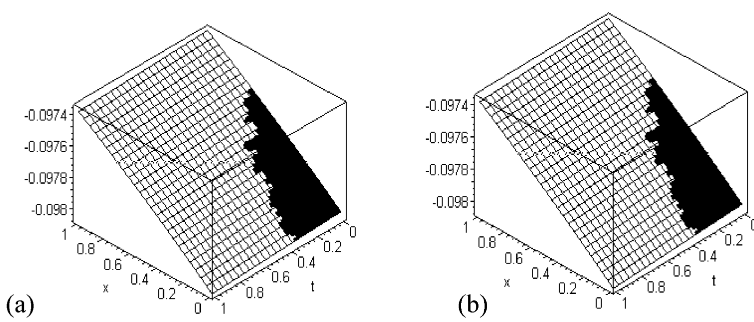

Fig. 3: The surface shows the solution u(x, t) when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 20$ (a) exact solution (b) approximate solution

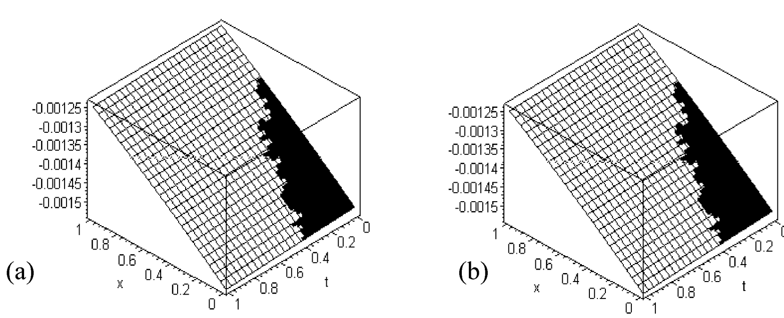

Fig. 4: The surface shows the solution v(x, t) when $k = 0.1$, $\lambda = 0.005$, $a = 0$, $\beta = 0.5$ and $x_0 = 20$ (a) exact solution (b) approximate solution

CONCLUSION

In this paper, we applied modified variational iteration method (MVIM) to find travelling wave solutions of Whitham–Broer–Kaup (WBK) equation. The proposed modification is made by introducing He's polynomials in the correction functional of the VIM. The use of Lagrange multiplier coupled with He's polynomials are the clear advantages of this technique over the decomposition method. Numerical results explicitly reveal the reliability of MVIM.

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