

## Generalized Multi-Phase Multivariate Regression Estimator for Partial Information Case Using Multi-Auxiliary Variables

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**Abstract:** In this paper we propose generalized multi-phase multivariate regression estimator in the presence of multi-auxiliary variables for estimating population mean vector of variables of interest. Some special cases have been deduced from the suggested estimator in the form of remarks. The expressions for mean square errors of proposed estimator and its special cases have also been derived and empirical study has also been carried out.

**Key words:** Multi-phase sampling . multivariate regression estimator . multi-auxiliary variables

### INTRODUCTION

The estimation of the population mean is an unrelenting issue in sampling theory and several efforts have been made to improve the precision of the estimators in the presence of multi-auxiliary variables. A variety of estimators have been proposed following different ideas of ratio, regression and product estimators.

Olkin [1] was the first author to deal with the problem of estimating the mean of a survey variable when information on several auxiliary variables is made available. Analogously to Olkin; Singh [2] gave a multivariate expression of Murthy's [3] product estimator, while Raj [4] put forward a method for using multi-auxiliary variables through a linear combination of single difference estimators. Moreover, Shukla [5] suggested a multiple regression estimator.

Singh and Namjoshi [6] discussed a class of multivariate regression estimators of population mean of study variable in two-phase sampling. Robinson [7] proposed a regression estimator ignoring some of the assumptions usually adopted in the literature Srivastava [8].

Agarwal *et al.* [9], moving away from Raj [4], illustrated a new approach to form a multivariate difference estimator which does not require the knowledge of any population parameter. Ahmed [10] put forward chain based general estimators using multivariate auxiliary information under multiphase sampling, while Kadilar and Cingi [11, 12] analyzed combinations of regression type estimators in the case of two auxiliary variables. Pradhan [13] suggested a chain regression estimator for two-phase sampling

using three auxiliary variables when the population mean of one auxiliary variable is unknown and other is known. Hanif *et al.* [14] suggested generalized multivariate ratio estimators in the presence of multi-auxiliary variables for estimating population mean of a study variable for multi-phase sampling. The estimators were proposed for both cases when information on all auxiliary variables is known (full information case) and unknown (no information case).

Ahmad *et al.* [15] proposed generalized regression-cum-ratio estimators for two phase sampling using multi-auxiliary variables for estimating the population mean of study variable. Ahmad *et al.* [16] developed generalized regression-in-regression estimators for two-phase sampling using multi-auxiliary variables for estimating the population mean of variable of interest.

Ahmad *et al.* [17] suggested generalized multi-phase multivariate ratio estimators for partial information case using multi-auxiliary variables. Ahmad *et al.* [18] developed multivariate regression estimators for full and no information cases in the presence of multi-auxiliary variables for estimating population mean of a study variable for multi-phase sampling.

In multipurpose surveys, the problem is to estimate population means of several variables simultaneously Swain [19]. Tripathi and Khattree [20] estimated means of several variables of interest, using multi-auxiliary variables for simple random sampling. Further Tripathi [21] extended the results to the case of two phase sampling.

We suggest generalized multivariate regression estimator for estimating a vector of population means of study variables for multi-phase sampling using

multi-auxiliary variables when information on some multi-auxiliary variables (Partial Information Case) is available for population [22].

Before suggesting the estimators we provide multi-phase sampling scheme and some useful notations and results in the following section.

### MULTI-PHASE SAMPLING USING MULTI-AUXILIARY VARIABLES

Consider a population of size  $N$  units. Let  $Y_1, Y_2, \dots, Y_p$  are  $p$  variables of interest and  $X_1, X_2, \dots, X_q$  are  $q$  auxiliary variables. For multi-phase sampling design let  $n_h$  and  $n_k$  ( $n_h < n_k$ ) be sample sizes for  $h^{\text{th}}$  and  $k^{\text{th}}$  phase respectively,  $x_{(h)i}$  and  $x_{(k)i}$  denote the  $i^{\text{th}}$  auxiliary variables from  $h^{\text{th}}$  and  $k^{\text{th}}$  phase samples respectively and  $y_{(k)i}$  denote  $i^{\text{th}}$  study variable from  $k^{\text{th}}$  phase sample. Let  $\bar{X}_i$ ,  $C_{x_i}$ ,  $C_y$ ,  $\rho_{y,x_i}$ ,  $\rho_{y,y_j}$  and  $\rho_{x_i,x_j}$  denote the population mean, coefficient of variation of  $i^{\text{th}}$  auxiliary variable, coefficient of variation of  $i^{\text{th}}$  variable of interest, correlation coefficient of  $i^{\text{th}}$  variable of interest and  $i^{\text{th}}$  auxiliary variable, correlation coefficient of  $i^{\text{th}}$  and  $j^{\text{th}}$  variable of interest and correlation coefficient of  $i^{\text{th}}$  and  $j^{\text{th}}$  auxiliary variables respectively.

Further let

$$\theta_h = \frac{1}{n_h} - \frac{1}{N}$$

and

$$\theta_k = \frac{1}{n_k} - \frac{1}{N}$$

are sampling fractions for  $h^{\text{th}}$  and  $k^{\text{th}}$  phase respectively. Also

$$y_{(k)i} = Y + e_{y_{(k)i}}, x_{(h)i} = X_i + e_{x_{(h)i}}$$

and

$$x_{(k)i} = X_i + e_{x_{(k)i}} \quad (i=1,2,\dots,k)$$

where  $e_{y_{(k)i}}$ ,  $e_{x_{(h)i}}$  and  $e_{x_{(k)i}}$  are sampling errors. We assume that

$$E_k(e_{y_{(k)i}}) = E_h(e_{x_{(h)i}}) = E_k(e_{x_{(k)i}}) = 0$$

where  $E_h$  and  $E_k$  denote the expectations of errors of  $h^{\text{th}}$  and  $k^{\text{th}}$  phase sampling. Then for simple random sampling without replacement for both first and second phases, we write by using phase wise operation of expectations as:

$$\begin{aligned} E_k(e_{y_{(k)i}})^2 &= \left(1 - \frac{n_k}{N}\right) \sigma_{y_{(k)i}}^2 \\ E_k(\bar{e}_{y_{(k)i}})^2 &= \left(1 - \frac{n_k}{N}\right) \frac{\sigma_{y_{(k)i}}^2}{n_k} = \theta_k \bar{Y}_i^2 C_{y_i}^2 \\ E_k(e_{y_{(k)i}} e_{x_{(h)i}}) &= \left(1 - \frac{n_k}{N}\right) \sigma_{y_{(k)i} x_{(h)i}} = \theta_k \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_{(k)i} x_{(h)i}} \\ E_k(\bar{e}_{y_{(k)i}} \bar{e}_{x_{(h)i}}) &= \left(1 - \frac{n_k}{N}\right) \frac{\sigma_{y_{(k)i} x_{(h)i}}}{n_k} = \theta_k \bar{Y}_i \bar{X}_i C_{y_i} C_{x_i} \rho_{y_{(k)i} x_{(h)i}} \\ E_h E_{\frac{1}{n_h}} \left[ e_{y_{(k)i}} (e_{x_{(h)i}} - e_{x_{(h)i}}) \right] &= E_h E_{\frac{1}{n_h}} (e_{y_{(k)i}} e_{x_{(h)i}}) - E_k(e_{y_{(k)i}} e_{x_{(h)i}}) \\ &= \frac{1}{N} (n_k - n_h) \sigma_{y_{(k)i} x_{(h)i}} \\ E_h E_{\frac{1}{n_h}} \left[ \bar{e}_{y_{(k)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) \right] &= \left(1 - \frac{n_h}{N}\right) \frac{\sigma_{y_{(k)i} x_{(h)i}}}{n_h} - \left(1 - \frac{n_k}{N}\right) \frac{\sigma_{y_{(k)i} x_{(h)i}}}{n_k} \\ &= (\theta_h - \theta_k) \bar{Y} \bar{X}_i C_{y_i} C_{x_i} \rho_{y_{(k)i} x_{(h)i}} \end{aligned}$$

Similarly

$$\begin{aligned} E_h E_{\frac{1}{n_h}} \left[ \bar{e}_{x_{(h)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) \right] &= (\theta_h - \theta_k) \sigma_{x_i}^2 \\ = (\theta_h - \theta_k) \bar{X}_i^2 C_{x_i}^2, E_h E_{\frac{1}{n_h}} \left[ \bar{e}_{x_{(h)i}} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) \right] &= 0 \\ E_h E_{\frac{1}{n_h}} \left( \bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}} \right)^2 &= (\theta_k - \theta_h) \sigma_{x_i}^2 = (\theta_k - \theta_h) \bar{X}_i^2 C_{x_i}^2 \\ E_h E_{\frac{1}{n_h}} \left[ (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) \right] &= (\theta_k - \theta_h) \sigma_{x_i x_j} \\ = (\theta_k - \theta_h) \bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j}, (i \neq j) & \end{aligned}$$

and

$$\begin{aligned} E_h E_{\frac{1}{n_h}} \left[ (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(h)i}}) \right] &= (\theta_h - \theta_k) \sigma_{x_i x_j} \\ = (\theta_h - \theta_k) \bar{X}_i \bar{X}_j C_{x_i} C_{x_j} \rho_{x_i x_j}, (i \neq j) & \end{aligned}$$

The following notations will be used in deriving the mean square errors of proposed estimators

$ R _{x_i}$	Determinant of population correlation matrix of variables $y$ , $X_1, X_2, \dots, X_{q-1}$ and $X_q$ .
$ R _{x_i}$	Determinant of $i^{\text{th}}$ minor of $ R _{x_i}$ corresponding to the $i^{\text{th}}$ element of $\rho_{yx_i}$ .
$\rho_{y,x_i}^2$	Denotes the multiple coefficient of determination of $y$ on $x_1, x_2, \dots, x_{i-1}$ and $x_i$ .
$\rho_{y,x_i}^2$	Denotes the multiple coefficient of determination of $y$ on $x_1, x_2, \dots, x_{q-1}$ and $x_q$ .
$ R _{x_{i+1}}$	Determinant of population correlation matrix of variables $x_{i+1}, X_{i+2}, \dots, X_{q-1}$ and $X_q$ .
$ R _{x_{i+1}}$	Determinant of population correlation matrix of variables $x_1, x_2, \dots, x_{q-1}$ and $x_q$ .
$ R _{y,x_i}$	Determinant of the correlation matrix of $y, x_1, x_2, \dots, x_{i-1}$ and $x_i$ .
$ R _{y,x_i}$	Determinant of the correlation matrix of $y, x_1, x_2, \dots, x_{q-1}$ and $x_q$ .

$ R _{y_i y_j}$	Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, \dots, x_r$ and $x_s$ , for $(i \neq j)$ .
$ R _{y_i y_j y_q}$	Determinant of the minor corresponding to $\rho_{y_i y_j}$ of the correlation matrix of $y_i, y_j, x_1, x_2, \dots, x_{q-1}$ and $x_q$ , for $(i \neq j)$ .

**Result: 1:** The following result will help in deriving the mean square errors of suggested estimators [23].

$$\frac{|R|_{y_i y_q}}{|R|_{x_q}} = (1 - \rho_{y_i x_q}^2)$$

### GENERALIZED MULTI-PHASE MULTIVARIATE REGRESSION ESTIMATOR FOR PARTIAL INFORMATION CASE

Let we have  $q$  auxiliary variable  $X_1, X_2, \dots, X_q$  and population means for first  $r$  auxiliary is not known and for the rest  $q-r=s$  is known. Let  $\bar{Y}_{(k)i}$  denotes the sample mean of  $i^{\text{th}}$  study variable from  $k^{\text{th}}$  phase and  $\bar{x}_{(h)i}$  and  $\bar{x}_{(k)i}$  denotes the sample mean of  $i^{\text{th}}$  auxiliary variable from  $h^{\text{th}}$  and  $k^{\text{th}}$  phase respectively. The generalized multi-phase multivariate regression estimator for estimating the mean vector of  $p$  variables of interest in the presence of  $q$  auxiliary variables for partial information case is suggested as:

$$T_{h \neq k p} = \begin{bmatrix} \bar{Y}_{(k)1} + \sum_{i=1}^r \alpha_{i1} (\bar{x}_{(h)i} - \bar{x}_{(k)i}) + \sum_{i=r+1}^{r+s} \beta_{i1} (\bar{X}_i - \bar{x}_{(h)i}) \\ + \sum_{i=r+1}^{r+s} \gamma_{i1} (\bar{X}_i - \bar{x}_{(k)i}) \bar{Y}_{(k)2} + \sum_{i=1}^r \alpha_{i2} (\bar{x}_{(h)i} - \bar{x}_{(k)i}) \\ + \sum_{i=r+1}^{r+s} \beta_{i2} (\bar{X}_i - \bar{x}_{(h)i}) + \sum_{i=r+1}^{r+s} \gamma_{i2} (\bar{X}_i - \bar{x}_{(k)i}) \\ \dots \bar{Y}_{(k)p} + \sum_{i=1}^r \alpha_{ip} (\bar{x}_{(h)i} - \bar{x}_{(k)i}) \\ + \sum_{i=r+1}^{r+s} \beta_{ip} (\bar{X}_i - \bar{x}_{(h)i}) + \sum_{i=r+1}^{r+s} \gamma_p (\bar{X}_i - \bar{x}_{(k)i}) \end{bmatrix} \quad (3.1)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are unknown constants, the expressions for these constants can be obtained for which elements of variance covariance matrix of estimator suggested in (3.1) will be minimum. We write (3.1) as:

$$T_{h \neq k p} = \begin{bmatrix} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) + \sum_{i=1}^r \alpha_{i1} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s} \beta_{i1} \bar{e}_{x_{(h)i}} \\ - \sum_{i=r+1}^{r+s} \gamma_{i1} \bar{e}_{x_{(k)i}} \dots (\bar{Y}_p + \bar{e}_{y_{(k)p}}) + \sum_{i=1}^r \frac{\alpha_{ip}}{\bar{X}_i} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \\ - \sum_{i=r+1}^{r+s} \beta_{ip} \bar{e}_{x_{(h)i}} - \sum_{i=r+1}^{r+s} \gamma_{ip} \bar{e}_{x_{(k)i}} \end{bmatrix}$$

or

$$T_{h \neq k p} = \left[ \begin{array}{cccc} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \dots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \\ + \left[ \begin{array}{cccc} \sum_{i=1}^r \alpha_{i1} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s} \beta_{i1} \bar{e}_{x_{(h)i}} - \sum_{i=r+1}^{r+s} \gamma_{i1} \bar{e}_{x_{(k)i}} \\ \sum_{i=1}^r \alpha_{i2} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s} \beta_{i2} \bar{e}_{x_{(h)i}} - \sum_{i=r+1}^{r+s} \gamma_{i2} \bar{e}_{x_{(k)i}} \\ \dots \sum_{i=1}^r \alpha_{ip} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) - \sum_{i=r+1}^{r+s} \beta_{ip} \bar{e}_{x_{(h)i}} + \sum_{i=r+1}^{r+s} \gamma_{ip} \bar{e}_{x_{(k)i}} \end{array} \right] \end{array} \right]$$

or

$$T_{h \neq k p} = \left[ \begin{array}{cccc} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \dots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \\ + \left[ \begin{array}{cccc} \sum_{i=1}^r \alpha_{i1} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) & \sum_{i=1}^r \alpha_{i2} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \\ \dots \sum_{i=1}^r \alpha_{ip} (\bar{e}_{x_{(h)i}} - \bar{e}_{x_{(k)i}}) \end{array} \right] \\ - \left[ \begin{array}{cccc} \sum_{i=r+1}^{r+s} \beta_{i1} \bar{e}_{x_{(h)i}} & \sum_{i=r+1}^{r+s} \beta_{i2} \bar{e}_{x_{(h)i}} & \dots & \sum_{i=r+1}^{r+s} \beta_{ip} \bar{e}_{x_{(h)i}} \end{array} \right] \\ - \left[ \begin{array}{cccc} \sum_{i=r+1}^{r+s} \gamma_{i1} \bar{e}_{x_{(k)i}} & \sum_{i=r+1}^{r+s} \gamma_{i2} \bar{e}_{x_{(k)i}} & \dots & \sum_{i=r+1}^{r+s} \gamma_{ip} \bar{e}_{x_{(k)i}} \end{array} \right] \end{array} \right]$$

or

$$T_{h \neq k p} = \begin{bmatrix} (\bar{Y}_1 + \bar{e}_{y_{(k)1}}) & (\bar{Y}_2 + \bar{e}_{y_{(k)2}}) & \dots & (\bar{Y}_p + \bar{e}_{y_{(k)p}}) \\ + [\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}] & [\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}] & \dots & [\bar{e}_{x_{(h)r}} - \bar{e}_{x_{(k)r}}] \\ [\alpha_{ij}]_{(r \times p)} - [\bar{e}_{x_{(h)r+1}} \bar{e}_{x_{(h)r+2}} \dots \bar{e}_{x_{(h)r+s}}]_{1 \times s} [\beta_{ij}]_{(s \times p)} \\ - [\bar{e}_{x_{(k)r+1}} \bar{e}_{x_{(k)r+2}} \dots \bar{e}_{x_{(k)r+s}}]_{1 \times s} [\gamma_{ij}]_{(s \times p)} \end{bmatrix}$$

or

$$T_{h \neq k p} = \bar{Y}_{1 \times p} + \bar{d}_{x_{(h)r}} A_{r \times p} - \bar{d}_{x_{(k)r}} B_{s \times p} - \bar{d}_{x_{(k)s}} C_{s \times p} \quad (3.2)$$

Where

$$\begin{aligned} \bar{d}_{x_{hk}} &= \left[ (\bar{X}_{(h)1} - \bar{X}_{(k)1}) \quad (\bar{X}_{(h)2} - \bar{X}_{(k)2}) \quad \dots \quad (\bar{X}_{(h)r} - \bar{X}_{(k)r}) \right] \\ &= \left[ (\bar{e}_{x_{(h)1}} - \bar{e}_{x_{(k)1}}) \quad (\bar{e}_{x_{(h)2}} - \bar{e}_{x_{(k)2}}) \quad \dots \quad (\bar{e}_{x_{(h)r}} - \bar{e}_{x_{(k)r}}) \right]_{1 \times r} \\ \bar{d}_{x_h} &= \left[ (\bar{X}_{(h)r+1} - \bar{X}_{r+1}) \quad (\bar{X}_{(h)r+2} - \bar{X}_{r+2}) \quad \dots \quad (\bar{X}_{(h)r+s} - \bar{X}_{r+s}) \right] \\ &= \left[ \bar{e}_{x_{(h)r+1}} \bar{e}_{x_{(h)r+2}} \dots \bar{e}_{x_{(h)r+s}} \right]_{1 \times s} \\ \bar{d}_{x_k} &= \left[ (\bar{X}_{(k)r+1} - \bar{X}_{r+1}) \quad (\bar{X}_{(k)r+2} - \bar{X}_{r+2}) \quad \dots \quad (\bar{X}_{(k)r+s} - \bar{X}_{r+s}) \right] \\ &= \left[ \bar{e}_{x_{(k)r+1}} \bar{e}_{x_{(k)r+2}} \dots \bar{e}_{x_{(k)r+s}} \right]_{1 \times s} \end{aligned}$$

$A = [\alpha_{ij}]_{(r \times p)}$ , for  $i = 1, 2, 3, \dots, r$  and  $j = 1, 2, 3, \dots, p$   
 $B = [\beta_{ij}]_{(s \times p)}$  and  $C = [\gamma_{ij}]_{(s \times p)}$ ; for  $i = r+1, r+2, r+3, \dots, r+s$  and  $j = 1, 2, 3, \dots, p$ . Letting,  $\bar{y} = \bar{Y} + \bar{d}_y$ ,

where

$$\bar{Y} = [\bar{Y}_1 \quad \bar{Y}_2 \quad \dots \quad \bar{Y}_p]$$

and

$$\bar{d}_y = \begin{bmatrix} \bar{e}_{y_{(k)1}} & \bar{e}_{y_{(k)2}} & \dots & \bar{e}_{y_{(k)p}} \end{bmatrix}$$

We write (3.2) as:

$$T_{hk(p \times p)} = \bar{Y} + \bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C.$$

We use information related to auxiliary variables from first and second phase both then the mean square error of  $T_{hk(1 \times p)}$  can be written as:

$$\begin{aligned} \Sigma_{T_{hk}(p \times p)} &= E_h E_{y \mid h} (T_{hk} - \bar{Y})' (T_{hk} - \bar{Y}) \\ &= E_h E_{y \mid h} \left[ \begin{bmatrix} (\bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C)' \\ (\bar{d}_y + \bar{d}_{x_{hk}} A - \bar{d}_{x_h} B - \bar{d}_{x_k} C) \end{bmatrix} \right] \quad (3.3) \end{aligned}$$

We can write

$$\begin{aligned} E_h E_{y \mid h} (\bar{d}_y \bar{d}_y') &= \theta_k \Sigma_y = \theta_k [\sigma_{y_i y_j}]_{(p \times p)}, \text{ for } i=j, \sigma_{y_i y_j} = \sigma_{y_i}^2 \\ E_h E_{y \mid h} (\bar{d}_y d_{x_h}) &= \theta_h \Sigma_{yx} = \theta_h [\sigma_{y_i x_j}]_{(p \times s)} \\ E_h E_{y \mid h} (\bar{d}_y d_{x_k}) &= \theta_k \Sigma_{yx} = \theta_k [\sigma_{y_i x_j}]_{(p \times r)} \\ E_h E_{y \mid h} (\bar{d}_y d_{x_{hk}}) &= (\theta_k - \theta_h) \Sigma_{yx} = (\theta_k - \theta_h) [\sigma_{y_i x_j}]_{(p \times r)} \\ E_h E_{y \mid h} (d_{x_h}' d_{x_h}') &= \theta_h \Sigma_x = \theta_h [\sigma_{x_i x_j}]_{(s \times s)}, \text{ for } i=j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{y \mid h} (d_{x_k}' d_{x_k}') &= \theta_k \Sigma_x = \theta_k [\sigma_{x_i x_j}]_{(s \times s)}, \text{ for } i=j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{y \mid h} (d_{x_{hk}}' d_{x_{hk}}') &= (\theta_k - \theta_h) \Sigma_x = (\theta_k - \theta_h) [\sigma_{x_i x_j}]_{(r \times r)}, \\ &\text{for } i=j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{y \mid h} (d_{x_h}' d_{x_k}) &= \theta_h \Sigma_x = \theta_h [\sigma_{x_i x_j}]_{(s \times s)}, \text{ for } i=j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \\ E_h E_{y \mid h} (d_{x_h}' d_{x_{hk}}) &= 0 \\ E_h E_{y \mid h} (d_{x_k}' d_{x_{hk}}) &= (\theta_h - \theta_k) \Sigma_x = (\theta_h - \theta_k) [\sigma_{x_i x_j}]_{(s \times r)}, \\ &\text{for } i=j, \sigma_{x_i x_j} = \sigma_{x_i}^2 \end{aligned}$$

Using above substitutions in expression of variance covariance matrix given in (3.3), we write:

$$\begin{aligned} \Sigma_{T_{hk}(p \times p)} &= \theta_k \Sigma_{y(p \times p)} - \theta_h \Sigma_{yx(p \times s)} A_{(s \times p)} - \theta_k \Sigma_{yx(p \times s)} B_{(s \times p)} \\ &+ (\theta_h - \theta_k) \Sigma_{yx(p \times s)} C_{(r \times p)} - \theta_l B'_{(p \times s)} \Sigma'_{yx(p \times s)} \\ &+ \theta_l B'_{(p \times s)} \Sigma_{x(s \times s)} B_{(s \times p)} + \theta_h B'_{(p \times s)} \Sigma_{x(s \times s)} C_{(s \times p)} \\ &- \theta_k C'_{(p \times s)} \Sigma'_{yx(p \times p)} + \theta_l C'_{(p \times s)} \Sigma_{x(s \times s)} B_{(s \times p)} \\ &+ \theta_k C'_{(p \times s)} \Sigma_{x(s \times s)} C_{(s \times p)} - (\theta_h - \theta_k) C_{(p \times s)} \Sigma_{x(s \times s)} A_{(r \times p)} \\ &+ (\theta_h - \theta_k) A_{(p \times r)} \Sigma'_{yx(p \times p)} - (\theta_h - \theta_k) A'_{(p \times r)} \Sigma'_{x(s \times s)} C_{(s \times p)} \\ &+ (\theta_k - \theta_h) A'_{(p \times r)} \Sigma_{x(s \times s)} A_{(r \times p)} \quad (3.4) \end{aligned}$$

The optimum values of unknown matrices A, B and C can be written as:

$$A_{(r \times p)} = W_{x(r \times r)}^{-1} \left( \Sigma'_{yx(p \times p)} - \Sigma_{x(s \times s)} \Sigma_{x(s \times s)}^{-1} \Sigma'_{yx(p \times p)} \right) \quad (3.5)$$

$$\begin{aligned} B_{(s \times p)} &= \left( \Sigma_{x(s \times s)}^{-1} \Sigma_{x(r \times r)} W_{x(r \times r)}^{-1} \Sigma'_{yx(p \times p)} \right) \\ &- \left( \Sigma_{x(s \times s)}^{-1} \Sigma_{x(s \times s)} W_{x(s \times s)}^{-1} \Sigma_{x(s \times s)} \Sigma_{x(s \times s)}^{-1} \Sigma'_{yx(p \times p)} \right) \quad (3.6) \end{aligned}$$

and

$$\begin{aligned} C_{(s \times p)} &= \left( \Sigma_{x(s \times s)}^{-1} \Sigma'_{yx(p \times p)} + \Sigma_{x(s \times s)}^{-1} \Sigma_{x(r \times r)} W_{x(r \times r)}^{-1} \Sigma_{x(r \times r)} \Sigma_{x(r \times r)}^{-1} \Sigma'_{yx(p \times p)} \right) \\ &- \left( \Sigma_{x(s \times s)}^{-1} \Sigma_{x(r \times r)} W_{x(r \times r)}^{-1} \Sigma'_{yx(p \times p)} \right) \quad (3.7) \end{aligned}$$

Using the above values of unknown matrices in the expression of mean square error given in (3.4), we write:

$$\begin{aligned} \Sigma_{T_{hk}(p \times p)} &= \theta_k \left( \Sigma_{y(p \times p)} - \Sigma_{yx(p \times s)} \Sigma_{x(s \times s)}^{-1} \Sigma'_{yx(p \times p)} \right) \\ &- (\theta_k - \theta_h) \left( \Sigma_{yx(p \times r)} - \Sigma_{yx(p \times s)} \Sigma_{x(s \times s)}^{-1} \Sigma_{x(r \times r)} \right) \quad (3.8) \\ &W_{x(r \times r)}^{-1} \left( \Sigma'_{yx(p \times p)} - \Sigma_{x(r \times r)} \Sigma_{x(r \times r)}^{-1} \Sigma'_{yx(p \times p)} \right) \end{aligned}$$

where

$$W_{x(r \times r)}^{-1} = \left( \Sigma_{x(r \times r)} - \Sigma_{x(r \times r)} \Sigma_{x(r \times r)}^{-1} \Sigma_{x(r \times r)} \right)^{-1}$$

provided  $\Sigma_{x(r \times r)}^{-1}$ ,  $W_{x(r \times r)}^{-1}$  and  $\Sigma_{x(s \times s)}^{-1}$  exist.

The variance covariance matrix in the form of variance of  $y_i$ , covariances and correlation coefficients of  $x_i$  and  $y_i$  can be written as:

$$\begin{aligned} \Sigma_{T_{hk}(p \times p)} &= \left[ \sigma_{y_i y_j} \begin{cases} \theta_k (\rho_{y_i y_j} - \rho_{y_i y_s} \rho_{y_s y_j}) + \\ \theta_h (\rho_{y_i y_j} - \rho_{y_i y_s} \rho_{y_s y_j}) \end{cases} \right]_{p \times p}; \\ &(i, j = 1, 2, \dots, p) \quad (3.9) \end{aligned}$$

for

$$i=j, \sigma_{y_i} \sigma_{y_j} = \sigma_{y_i}^2, \rho_{y_i y_j} = 1, \rho_{y_i y_s} = \rho_{y_s y_j}^2$$

and

$$\rho_{y_i y_s} = \rho_{y_s y_j}^2$$

In determinants of correlation matrices for  $|R|_{x_s} \neq 0$  and  $|R|_{x_s} \neq 0$ , (3.9) can be written as:

$$\begin{aligned} \Sigma_{T_{hk}(p \times p)} &= \left[ \sigma_{y_i} \sigma_{y_j} \left\{ \theta_k \frac{|R|_{y_i y_j x_s}}{|R|_{x_s}} + \theta_h \left( \frac{|R|_{y_i y_j x_s}}{|R|_{x_s}} - \frac{|R|_{y_i y_s x_s}}{|R|_{x_s}} \right) \right\} \right]_{p \times p} \\ &(i, j = 1, 2, \dots, p) \quad (3.10) \end{aligned}$$

for

$$i=j, \sigma_{y_i} \sigma_{y_j} = \sigma_{y_i}^2, |R|_{y_i y_j x_q} = |R|_{y_i x_q}$$

and

$$|R|_{y_i y_j x_s} = |R|_{y_i x_s}$$

**Remark-1:** To develop generalized multivariate regression estimator for two-phase sampling using multi-auxiliary variables for Partial Information Case, replace h by 1 and k by 2 in (3.1), we get the following estimator,

$$\begin{aligned} T_{1 \neq k p} = & \left[ \bar{y}_{(2)1} + \sum_{i=1}^r \alpha_{11} (\bar{x}_{(1)i} - \bar{x}_{(2)i}) + \sum_{i=r+1}^{r+s+q} \beta_{11} (\bar{X}_i - \bar{x}_{(1)i}) \right. \\ & + \sum_{i=r+1}^{r+s+q} \gamma_{11} (\bar{X}_i - \bar{x}_{(2)i}) \\ & \bar{y}_{(2)2} + \sum_{i=1}^r \alpha_{12} (\bar{x}_{(1)i} - \bar{x}_{(2)i}) + \sum_{i=r+1}^{r+s+q} \beta_{12} (\bar{X}_i - \bar{x}_{(1)i}) \\ & \left. + \sum_{i=r+1}^{r+s+q} \gamma_{12} (\bar{X}_i - \bar{x}_{(2)i}) \right] \\ & \dots \bar{y}_{(2)p} + \sum_{i=1}^r \alpha_{1p} (\bar{x}_{(1)i} - \bar{x}_{(2)i}) + \sum_{i=r+1}^{r+s+q} \beta_{1p} (\bar{X}_i - \bar{x}_{(1)i}) \\ & + \sum_{i=r+1}^{r+s+q} \gamma_{1p} (\bar{X}_i - \bar{x}_{(2)i}) \end{aligned} \quad (3.11)$$

The expressions of unknown matrices for which the mean square error of above estimator will be minimum and are same as given in (3.5), (3.6) and (3.7). The expression for variance covariance matrix can be directly written from (3.8) just replacing h by 1 and k by 2 as:

$$\begin{aligned} \Sigma_{T_{12} \neq p} = & \theta_2 \left( \Sigma_{y_{(pp)}} - \Sigma_{y_{(ps)}} \Sigma_{x_{(ss)}}^{-1} \Sigma'_{y_{(sp)}} \right) \\ & - (\theta_2 - \theta_1) \left( \Sigma_{y_{(ps)}} - \Sigma_{y_{(ps)}} \Sigma_{x_{(ss)}}^{-1} \Sigma_{x_{(ss)}} \right) \\ & W_{x_{(ss)}}^{-1} \left( \Sigma'_{y_{(sp)}} - \Sigma_{x_{(ss)}} \Sigma_{x_{(ss)}}^{-1} \Sigma'_{y_{(sp)}} \right) \end{aligned} \quad (3.12)$$

The variance covariance matrix in the form of variance of  $y_i$ , covariances and correlation coefficients of  $x_i$  and  $y_i$  is written as:

$$\begin{aligned} \Sigma_{T_{12} \neq p} = & \left[ \sigma_{y_i} \sigma_{y_j} \left\{ \begin{array}{l} \theta_2 (\rho_{y_i y_j} - \rho_{y_i y_j x_q}) \\ + \theta_1 (\rho_{y_i y_j x_q} - \rho_{y_i y_j x_s}) \end{array} \right\} \right]_{p \times p}; \\ & (i, j = 1, 2, \dots, p) \end{aligned} \quad (3.13)$$

for

$$i=j, \sigma_{y_i} \sigma_{y_j} = \sigma_{y_i}^2, \rho_{y_i y_j} = 1, \rho_{y_i y_j x_q} = \rho_{y_i x_q}^2$$

and

$$\rho_{y_i y_j x_s} = \rho_{y_i x_s}^2$$

In determinants of correlation matrices for  $|R|_{x_s} \neq 0$  and  $|R|_{x_q} \neq 0$ , (3.13) can be written as:

$$\Sigma_{T_{12}(pp)} = \left[ \sigma_{y_i} \sigma_{y_j} \left\{ \theta_2 \frac{|R|_{y_i y_j x_q}}{|R|_{x_q}} + \theta_1 \left( \frac{|R|_{y_i y_j x_s}}{|R|_{x_s}} - \frac{|R|_{y_i y_j x_q}}{|R|_{x_q}} \right) \right\} \right] \quad (3.14)$$

(i, j = 1, 2, ..., p)

for

$$i=j, \sigma_{y_i} \sigma_{y_j} = \sigma_{y_i}^2, |R|_{y_i y_j x_q} = |R|_{y_i x_q}$$

and

$$|R|_{y_i y_j x_s} = |R|_{y_i x_s}$$

**Remark-2:** We can develop a univariate generalized regression estimator for multiphase sampling using multi auxiliary variable for Partial Information Case if we put p = 1 in (3.1) as:

$$\begin{aligned} T_{h \neq k p} = & \bar{y}_{(k)1} + \sum_{i=1}^r \alpha_i (\bar{x}_{(h)i} - \bar{x}_{(k)i}) + \sum_{i=r+1}^{r+s+q} \beta_i (\bar{X}_i - \bar{x}_{(h)i}) \\ & + \sum_{i=r+1}^{r+s+q} \gamma_i (\bar{X}_i - \bar{x}_{(k)i}) \end{aligned} \quad (3.15)$$

The expression for vectors of unknown constants for which the mean square error will be minimum can be written from (3.5), (3.6) and (3.7) as

$$A_{(r \times 1)} = W_{x_{(rs)}}^{-1} \left( \Sigma'_{y_{(rs)}} - \Sigma_{x_{(rs)}} \Sigma_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} \right) \quad (3.16)$$

$$\begin{aligned} B_{(s \times 1)} = & \left( \Sigma_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} W_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} \right) \\ & - \left( \Sigma_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} W_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} \Sigma_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} \right) \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} C_{(s \times 1)} = & \left( \Sigma_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} + \Sigma_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} W_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} \Sigma_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} \right) \\ & - \left( \Sigma_{x_{(rs)}}^{-1} \Sigma_{x_{(rs)}} W_{x_{(rs)}}^{-1} \Sigma'_{y_{(rs)}} \right) \end{aligned} \quad (3.18)$$

The above expressions for unknown matrices can be written in determinants form as:

$$\alpha_i = (-1)^{i+1} \frac{C_y}{C_{x_i}} \frac{|R|_{y_{(i)}}}{|R|_{x_i}} = (-1)^{i+1} \frac{\bar{X}_i}{Y} \beta_{y_{(i)} x_i}, (i = 1, 2, \dots, r) \quad (3.19)$$

$$\begin{aligned}\beta_i &= (-1)^{i+1} \frac{C_y}{C_{x_i}} \left\{ \frac{\left| R_{yx_i} \right|_{yx_s}}{\left| R_{x_i} \right|_{x_s}} - \frac{\left| R_{yx_i} \right|_{yx_q}}{\left| R_{x_i} \right|_{x_q}} \right\} \\ &= (-1)^{i+1} \frac{\bar{X}_i}{Y} (\beta_{yx_{x_s}} - \beta_{yx_{x_q}}), \quad (i=r+1, r+2, \dots, r+s)\end{aligned}\quad (3.20)$$

$$\gamma_i = (-1)^{i+1} \frac{C_y}{C_{x_i}} \frac{\left| R_{yx_i} \right|_{yx_q}}{\left| R_{x_i} \right|_{x_q}} = (-1)^{i+1} \frac{\bar{X}_i}{Y} \beta_{yx_{x_q}} \quad (i=r+1, r+2, \dots, r+s) \quad (3.21)$$

The expression for mean square error can be directly written from (3.8) as:

$$\begin{aligned}MSE(T_{hk}) &= \theta_k \left( \sigma_y^2 - \sum_{y_{(k)s}} \Sigma_{x_{(s)}}^{-1} \Sigma'_{y_{(k)s}} \right) \\ &\quad - (\theta_k - \theta_h) \left\{ \left( \sum_{y_{(k)s}} - \sum_{y_{(k)s}} \Sigma_{x_{(s)}}^{-1} \Sigma_{x_{(s)}} \right) \right. \\ &\quad \left. W_{x_{(s)r}}^{-1} \left( \Sigma'_{y_{(k)s}} - \Sigma_{x_{(r)s}} \Sigma_{x_{(s)}}^{-1} \Sigma'_{y_{(r)s}} \right) \right\} \quad (3.22)\end{aligned}$$

It can be written the form of multiple coefficient of determination as:

$$MSE(T_{hk}) = \bar{Y}^2 C_y^2 \left[ \theta_k \left( 1 - \rho_{y_{x_q}}^2 \right) + \theta_h \left( \rho_{y_{x_q}}^2 - \rho_{y_{x_s}}^2 \right) \right] \quad (3.23)$$

**Remark-3:** To develop a generalized univariate regression estimator for two-phase sampling using multi-auxiliary variables for Partial Information Case we put  $h = 1$  and  $k = 2$  in (3.15). The required estimator becomes

$$\begin{aligned}T_{1 \notin k(p)} &= \bar{Y}_{(k)1} + \sum_{i=1}^r \alpha_i (\bar{X}_{(1)i} - \bar{X}_{(2)i}) + \sum_{i=r+1}^{r+s=q} \beta_i (\bar{X}_i - \bar{X}_{(1)i}) \\ &\quad + \sum_{i=r+1}^{r+s=q} \gamma_i (\bar{X}_i - \bar{X}_{(2)i})\end{aligned}\quad (3.24)$$

The expression for vectors of unknown constants for which the mean square error will be minimum are same as given in (3.16), (3.17) and (3.18) and these expressions are also given in determinants of correlation matrices in (3.19), (3.20) and (3.21). The expression for mean square error can be written from (3.22) just by replacing  $h = 1$  and  $k = 2$  as:

$$\begin{aligned}MSE(T_{12}) &= \theta_2 \left( \sigma_y^2 - \sum_{y_{(2)s}} \Sigma_{x_{(s)}}^{-1} \Sigma'_{y_{(2)s}} \right) \\ &\quad - (\theta_2 - \theta_1) \left\{ \left( \sum_{y_{(2)s}} - \sum_{y_{(2)s}} \Sigma_{x_{(s)}}^{-1} \Sigma_{x_{(s)}} \right) \right. \\ &\quad \left. W_{x_{(s)r}}^{-1} \left( \Sigma'_{y_{(2)s}} - \Sigma_{x_{(r)s}} \Sigma_{x_{(s)}}^{-1} \Sigma'_{y_{(r)s}} \right) \right\} \quad (3.25)\end{aligned}$$

It can be written the form of multiple coefficient of determination as:

$$MSE(T_{12}) = \bar{Y}^2 C_y^2 \left[ \theta_2 \left( 1 - \rho_{y_{x_q}}^2 \right) + \theta_1 \left( \rho_{y_{x_q}}^2 - \rho_{y_{x_s}}^2 \right) \right] \quad (3.26)$$

**Remark-4:** Generalized multivariate regression estimator as suggested by Hanif *et al.* (2009) for multiphase sampling using multi-auxiliary variables when information on all auxiliary variables is not available for population (No Information Case) can be developed by putting  $\beta_i$ 's and  $\gamma_i$ 's equals to zero in (3.1) as:

$$\begin{aligned}T'_{h \notin k(p)} &= \left[ \bar{Y}_{(k)1} + \sum_{i=1}^r \alpha_{il} (\bar{X}_{(h)i} - \bar{X}_{(k)i}) \right. \\ &\quad \left. \bar{Y}_{(k)2} + \sum_{i=1}^r \alpha_{l2} (\bar{X}_{(h)i} - \bar{X}_{(k)i}) \right. \\ &\quad \left. \dots \bar{Y}_{(k)p} + \sum_{i=1}^r \alpha_{lp} (\bar{X}_{(h)i} - \bar{X}_{(k)i}) \right]\end{aligned}\quad (3.27)$$

The expression of unknown matrix for which the mean square error will be minimum can be directly obtained by considering only those matrices from (3.5), (3.6) and (3.7) those includes only order  $p \times r$  and  $r \times r$  than we get the required matrix that is  $\Sigma_{x_{(r)s}}^{-1} \Sigma'_{y_{(r)s}}$ . The variance covariance matrix can be obtained from (3.8) just by considering those matrices having order  $p \times r$  and  $r \times r$ . The variance covariance matrix is

$$S_{T'_{hk}(p \times p)} = \theta_k \Sigma_{y_{(p \times p)}} - (\theta_k - \theta_h) \sum_{y \notin p \times r} \sum_{x \in (r \times r)} \sum_{y \notin x \times p} \quad (3.28)$$

All special cases of estimator given in (3.27), in the case on no information and full information, have been discussed by Hanif *et al.* (2009).

## EMPIRICAL STUDY

Obviously estimator for which the information on all auxiliary variables is available for population (full information case) will be more efficient than that for which the information on some auxiliary variables is available for population (partial information case). The estimator for partial information case will be efficient than the estimator for which the information on all auxiliary variables is not available for population (no information case). In the case of multiphase, the estimator will be less efficient by increasing the phases but cost effective. To illustrate the above statements, empirical study has been carried.

For the justification of above statements, the empirical study is carried out by using determinant of

## Appendix A

Table A-1.1: Detail of populations

Sr. #	Source of populations
1	Population census report of Jhang district (1998), Pakistan
2	Population census report of Gujrat district (1998), Pakistan
3	Population census report of Kasur (1998) Pakistan
4	Population census report of Sialkot district (1998), Pakistan

Table A-1.2: Description of variables (Each variables is taken from rural locality)

Description of variables	
$Y_1$	Literacy ratio
$Y_2$	Population of currently married
$Y_3$	Total household
$X_1$	Population of both sexes
$X_2$	Population of primary but below metric
$X_3$	Population of metric and above
$X_4$	Population of 18 years old and above
$X_5$	Population of women 15-49 years old

Table A-1.3: Parameters of populations for calculating the Matrices of MSE's of multivariate estimators and MSE's of univariate estimators

District	N	$n_1$	$n_2$	$n_3$	$n_4$	$n_5$	$\bar{Y}_1$	$\bar{Y}_2$	$\bar{Y}_3$	$C_{y_1}$	$C_{y_2}$	$C_{y_3}$
Jhang	368	184	92	46	23	12	29.705	860.11	897.71	0.270	0.595	0.512
Gujrat	204	102	51	26	13	6	57.535	1101.280	1102.540	0.145	0.484	0.487
Kasur	181	91	45	23	11	6	31.890	1393.200	1449.020	0.747	0.551	0.530
Sialkot	269	135	67	34	17	8	52.061	1058.740	998.220	0.147	0.647	0.646
District	$\sigma_{y_1}$	$\sigma_{y_2}$	$\sigma_{y_3}$	$\sigma_{x_1}$	$\sigma_{x_2}$	$\sigma_{x_3}$	$\sigma_{x_4}$	$\sigma_{x_5}$	$\rho_{y_1 y_2}$			
Jhang	8.022	511.908	459.842	5626.450	455.060	170.670	2455.170	1064.480	.182			
Gujrat	8.364	533.041	537.236	3507.160	940.480	381.690	8139.680	830.010	.055			
Kasur	23.823	767.636	767.796	5515.420	1095.690	357.890	2719.210	1355.640	.295			
Sialkot	7.641	685.019	644.886	4787.250	1172.710	603.220	2461.590	1151.320	.324			
District	$\rho_{y_1 y_3}$	$\rho_{y_2 y_3}$	$\rho_{y_1 x_1}$	$\rho_{y_1 x_2}$	$\rho_{y_1 x_3}$	$\rho_{y_1 x_4}$	$\rho_{y_1 x_5}$	$\rho_{y_2 x_1}$	$\rho_{y_2 x_2}$	$\rho_{y_2 x_3}$	$\rho_{y_2 x_4}$	$\rho_{y_2 x_5}$
Jhang	0.164	0.733	0.131	0.460	0.548	0.185	0.129	0.428	0.912	0.659	0.484	0.425
Gujrat	0.056	0.988	0.092	0.334	0.543	0.069	0.103	0.995	0.941	0.764	0.490	0.996
Kasur	0.301	0.989	0.299	0.255	0.352	0.301	0.250	0.998	0.758	0.879	0.989	0.799
Sialkot	0.316	0.997	0.323	0.426	0.461	0.338	0.313	0.999	0.983	0.931	0.996	0.939
District	$\rho_{y_3 x_1}$	$\rho_{y_3 x_2}$	$\rho_{y_3 x_3}$	$\rho_{y_3 x_4}$	$\rho_{y_3 x_5}$			$\rho_{x_1 x_2}$	$\rho_{x_1 x_3}$	$\rho_{x_1 x_4}$		
Jhang	0.474	0.732	0.748	0.559	0.489			0.416	0.421	0.317		
Gujrat	0.984	0.933	0.749	0.487	0.986			0.954	0.796	0.509		
Kasur	0.991	0.752	0.878	0.988	0.792			0.764	0.889	0.993		
Sialkot	0.996	0.980	0.933	0.994	0.938			0.983	0.931	0.997		
District	$\rho_{x_1 x_5}$	$\rho_{x_2 x_3}$	$\rho_{x_2 x_4}$	$\rho_{x_2 x_5}$	$\rho_{x_3 x_4}$	$\rho_{x_3 x_5}$	$\rho_{x_4 x_5}$	$\rho_{y_1 x_3}$	$\rho_{y_1 x_5}$	$\rho_{y_2 x_3}$	$\rho_{y_2 x_5}$	
Jhang	0.275	0.824	0.475	0.432	0.590	0.464	0.325	0.313	0.885			
Gujrat	0.996	0.892	0.500	0.958	0.420	0.797	0.505	0.992	0.996			
Kasur	0.802	0.798	0.764	0.614	0.896	0.719	0.797	0.979	0.995			
Sialkot	0.939	0.959	0.985	0.928	0.939	0.887	0.938	0.992	0.997			

Table A-1.4.1 Determinants of matrices of MSE's of multivariate Regression estimators for pair-wise phases (No Information Case)

District	$T_{12}$ (h=1,k=2)	$T_{13}$ (h=1,k=3)	$T_{14}$ (h=1,k=4)	$T_{15}$ (h=1,k=5)	$T_{23}$ (h=2,k=3)	$T_{24}$ (h=2,k=4)
Jhang	212279.97	1223241.603	7221747.657	46128460.86	3572866.63	16011134.82
Gujrat	95363.18	312081.54	1102996.85	4211396.91	1123210.67	3372979.29
Kasur	203091.03	901801.71	3838230.04	16192403.14	2176260.22	8973735.68
Sialkot	9555.27	41464.31	173806.26	723929.58	126175.45	482925.70
District	$T_{25}$ (h=2,k=5)	$T_{34}$ (h=3,k=4)	$T_{35}$ (h=3,k=5)	$T_{45}$ (h=4,k=5)		
Jhang	79961159.68	38868782.02	158093587.70	2.60491E+11		
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63		
Kasur	36805031.49	19922737.86	79389873.19	170075469.89		
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88		

Table A-1.4.2:Determinants of variance covariance matrices of multivariate Regression estimators for pair-wise phases (Partial Information Case)

District	$T_{12}$ (h=1,k=2)	$T_{13}$ (h=1,k=3)	$T_{14}$ (h=1,k=4)	$T_{15}$ (h=1,k=5)	$T_{23}$ (h=2,k=3)	$T_{24}$ (h=2,k=4)
Jhang	138163.6	890963.2	5820643.1	40379695.2	2250013.1	11146370.3
Gujrat	1683.1	10844.4	73847.7	534079.5	18218.9	103581.9
Kasur	247034.4	1500728.6	9605459.6	66188757.3	2573608.2	14114833.3
Sialkot	322.1	2207.0	15843.0	118825.5	3970.3	22574.0
District	$T_{25}$ (h=2,k=5)	$T_{34}$ (h=3,k=4)	$T_{35}$ (h=3,k=5)	$T_{45}$ (h=4,k=5)		
Jhang	61382546.8	24203288.5	108298411.4	222062545.3		
Gujrat	653434.5	167622.1	901358.8	1434473.5		
Kasur	84669409.0	23285985.9	121940138.0	197720243.3		
Sialkot	145036.8	38559.8	202554.0	338300.7		

Table A-1.4.3: Determinants of matrices of MSE's of multivariate Regression estimators for each phase (Full Information Case)

District	$T_1$ (k=1)	$T_2$ (k=2)	$T_3$ (k=3)	$T_4$ (k=4)	$T_5$ (k=5)
Jhang	1023.378901	27631.23032	351018.963	3453903.79	30487480.83
Gujrat	27.15981853	367.7474988	3710.595857	33124.8694	279522.8025
Kasur	103.7803005	1306.215104	12804.97189	112839.0944	946342.2579
Sialkot	2.156156072	36.85754751	405.2293669	3755.836241	32256.39965

Table A-1.5.1: MSE's of univariate Regression estimators for pair-wise phases (No Information Case)

District	$T_{12}$ (h=1,k=2)	$T_{13}$ (h=1,k=3)	$T_{14}$ (h=1,k=4)	$T_{15}$ (h=1,k=5)	$T_{23}$ (h=2,k=3)	$T_{24}$ (h=2,k=4)
Jhang	212279.9	1223241.6	7221747.6	46128460.8	3572866.6	16011134.8
Gujrat	95363.1	312081.5	1102996.8	4211396.9	1123210.6	3372979.2
Kasur	203091.0	901801.7	3838230.0	16192403.1	2176260.2	8973735.6
Sialkot	9555.27	41464.3	173806.26	723929.5	126175.4	482925.7
District	$T_{25}$ (h=2,k=5)	$T_{34}$ (h=3,k=4)	$T_{35}$ (h=3,k=5)	$T_{45}$ (h=4,k=5)		
Jhang	79961159.68	38868782.02	158093587.7	2.60491E+11		
Gujrat	11251537.17	10702183.50	30944428.16	93069748.63		
Kasur	36805031.49	19922737.86	79389873.19	170075469.89		
Sialkot	1897366.87	1256890.23	4554991.24	11150157.88		

Table A-1.5.2: MSE's of univariate Regression estimators for pair-wise phases (Partial Information Case)

District	$T_{12}$ (h=1,k=2)	$T_{13}$ (h=1,k=3)	$T_{14}$ (h=1,k=4)	$T_{15}$ (h=1,k=5)	$T_{23}$ (h=2,k=3)	$T_{24}$ (h=2,k=4)
Jhang	652.99	980.5526	1635.678	2945.928	1795.189	2450.314
Gujrat	309.4727	900.3699	2082.164	4445.753	624.3232	1806.117
Kasur	378.5487	1044.532	2376.499	5040.433	771.9548	2103.922
Sialkot	234.4814	682.5967	1578.827	3371.289	474.9672	1371.198
District	$T_{25}$ (h=2,k=5)	$T_{34}$ (h=3,k=4)	$T_{35}$ (h=3,k=5)	$T_{45}$ (h=4,k=5)		
Jhang	4079.586	5389.837	8648.382	4079.586		
Gujrat	1254.024	3617.613	2513.426	1254.024		
Kasur	1558.767	4222.701	3132.391	1558.767		
Sialkot	955.9386	2748.4	1917.882	955.9386		

Table A-1.5.3: MSE's of univariate Regression estimators for each phase (Full Information Case)

District	$T_1$ (k=1)	$T_2$ (k=2)	$T_3$ (k=3)	$T_4$ (k=4)	$T_5$ (k=5)
Jhang	81.89064	245.6719	573.2344	1228.36	2538.61
Gujrat	213.5579	509.0065	1099.904	2281.698	4645.286
Kasur	251.1011	584.0929	1250.076	2582.043	5245.977
Sialkot	142.167	366.2247	814.34	1710.571	3503.032

variance covariance matrices/MSE's of newly developed estimators for partial information case and those developed by Hanif *et al.* (2009) for no and full information cases. We consider four natural populations. The detail of populations and variables description is given in Table A-1.1 and Table A-1.2 respectively of Appendix A. We consider three variables of interests denoted by Y's and five auxiliary variables denoted by X's for computing the determinants of matrices of MSE's of multivariate Regression estimators and for univariate we consider  $Y_2$  as study variable and the same five auxiliary variables as considered in multivariate case. The necessary parameters of populations for computing MSE's are given in A-1.3. We calculate pair-wise determinant of variance covariance matrices/MSE's for no information case and for full information case we calculate variance covariance matrices /MSE's for each phase for first five phases. The determinant of variance covariance matrices of multivariate regression estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.4.1, for partial information case, in Table A-1.4.2 and using each phase for full information case in Table A-1.4.3. The mean square errors of univariate estimators for multiphase sampling using pair-wise phases for no information case are given in Table A-1.5.1 and for partial information case in Table A-1.5.3 and for full information case using each phase in A-1.5.3.

From Table A-1.4.1, Table A-1.4.2 and Table A-1.4.3, we can say that the multivariate Regression estimators for full information case are more efficient than partial information case and estimators for partial information case are more efficient than no information case for each phase e.g.  $T_2$  is more efficient than  $T_{12}$ ,  $T_3$  is more efficient than  $T_{13}$  &  $T_{23}$  etc. and the same is true for univariate regression estimators (see Table A-1.5.1, Table A-1.5.2 and Table A-1.5.3). Furthermore we can say for no information case and partial information case from Table A-1.4.1 and Table A-1.4.2 that as we increase phase the efficiency decreases e.g.  $T_{12}$ , is more efficient than all others,  $T_3$  is more efficient than all others except  $T_{12}$ ,  $T_{34}$  is more efficient than  $T_{35}$ ,  $T_{45}$  but less efficient than all others and so on, similarly the same argument can be made for univariate case given in Table A-1.5.1. Also for full information case the estimators become less efficient as we increase phases because the sample size decreases by increasing phases, it can be seen from Table A-1.4.2 and A-1.5.2 for multivariate and univariate estimators respectively.

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