World Applied Sciences Journal 10 (3): 327-329, 2010
ISSN 1818-4952

Coupled Witham-broer-Kaup Equations and Expansion Method

A.H. Mahbobi, M.Valizadeh and A. Roozi

Islamic Azad University of Gonbad Kavous Branch, P.O. Box 49718, P.C. 36796, Gonbad, Iran

Abstract: In this work, we construct the travelling wave solutions involving parameters of the coupled Witham-Broer-Kaup equations, by using the $(\frac{G'}{G})$ -expansion method. When the parameters are taken special values, the solitary waves are derived from the traveling waves. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions.

Key words: Coupled Witham · Broer · Kaup equations · $(\frac{G'}{G})$ -expansion method

INTRODUCTION

In recent years, searching for explicit solutions of nonlinear evolution equations by using various methods has become the main goal for many authors and many powerful methods to construct exact solutions of nonlinear evolution equations have been established and developed such as the tanh-function expansion and its various extension [1, 2], the Jacobi elliptic function expansion [3, 4]. Very recently, Wang *et al*. [5] introduced a new method called the $(\frac{G'}{G})$ -expansion method to look for travelling wave solutions of nonlinear evolution equations[2,7]. The $(\frac{G'}{G})$ -expansion method is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in $(\frac{G'}{G})$ and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation(ODE).

Description of the $(\frac{G'}{G})$ -expansion method: Considering the nonlinear partial differential equation in the form

$$P(u,u_x,u_t,u_{tt},u_{xt},u_{xx},\ldots) \tag{1}$$

Where u = u(x, t) is an unknown function, p is a polynomial in u = u(x, t) and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of the $(\frac{G'}{G})$ -expansion method.

Step 1: Combining the independent variables x and t into one variable $\xi = x-vt$, we suppose that

$$u(x,t) = u(\xi), \qquad \xi = x - vt \tag{2}$$

The travelling wave variable (2) permits us to reduce Eq(1) to an ODE for $G = G(\xi)$, namely

$$P(u, -vu',u',v^2u'', -vu'', u'',\ldots) = 0 \tag{3}$$

Step 2: Suppose that the solution of ODE (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = \alpha(\frac{G'}{G}) + \ldots, \tag{4}$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \tag{5}$$

$\alpha_m,\ldots,\lambda$ and μ are constants to be determined later $\alpha_m \neq 0$, the unwritten part in 4 is also a polynomial in $(\frac{G'}{G})$, but the degree of which is generally equal to or less than $m - 1$, the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3).

Corresponding Author: A.H. Mahbobi, Islamic Azad University of Gonbad Kavous Branch, P.O. Box 49718, P.C. 36796, Gonbad, Iran. E-mail: amirhossein.mahbobi@yahoo.com.

Step 3: By substituting (4) into Eq. (3) and using the second order linear ODE (5), collecting all terms with the same order $(\frac{G'}{G})$ together, the left-hand side of Eq. (3) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m,\ldots,\lambda$ and μ.

Step 4: Assuming that the constants $\alpha_m,\ldots,\lambda$ and μ can be obtained by solving the algebraic equations in Step 3, since the general solutions of the second order LODE (5) have been well known for us, then substituting $\alpha_m,\ldots,v$ and the general solutions of Eq. (5) into (4) we have more travelling wave solutions of the nonlinear evolution equation (1).

Cupled Witham-Broer-Kaup equations: In this section we consider the Cupled Witham-Broer-Kaup equations in the form.

$$\begin{cases} u + uu + v + \beta u = 0 \\ v + (uv) + \alpha u - \beta v = 0 \end{cases}$$

The travelling wave variable below

$$u(x,t) = u(\xi) \qquad \xi = x - vt \tag{7}$$

Permits us converting Eq.(7) into an ODE for $G = G(\xi)$

$$\begin{cases} -\omega u' + \frac{1}{2}(u)' + v' + \beta u'' = 0 \\ -\omega v' + (uv)' + \alpha u''' - \beta v'' = 0 \end{cases}$$

Integrating it with respect to ξ once yields

$$\begin{cases} -\omega u + \frac{1}{2}(u) + v + \beta u' + c = 0 & (6) \\ -\omega v + uv + \alpha u'' - \beta v' + c = 0 & (7) \end{cases}$$

For simplicity we consider c_1 equal to zero and by substituting equation (6) into equation (7) we have

$$-\omega u + \frac{3}{2}\omega u - \frac{1}{2}u + (\alpha + \beta)u'' + c = 0 \tag{8}$$

Suppose that the solution of ODE (8) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$u(\xi) = \alpha(\frac{G'}{G}) + \ldots,$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

α_1, α_0, v and μ are to be determined later.

By using (9) and (10) and considering the homogeneous balance between u'' and u^3 in Eq. (8) we required that $3m = m+2$ then $m = 1$. So we can write (9) as

$$u(\xi) = \alpha_1(\frac{G'}{G}) + \alpha_0 \tag{11}$$

Therefore

$$u^3 = \alpha_1^3(\frac{G'}{G})^3 + 3\alpha_1^2\alpha_0(\frac{G'}{G})^2 + 3\alpha_1\alpha_0^2(\frac{G'}{G}) + \alpha_0^3 \tag{12}$$

$$u^2 = \alpha_1^2(\frac{G'}{G})^2 + 2\alpha_1\alpha_0(\frac{G'}{G}) + \alpha_0 \tag{13}$$

By using (11) and (10) it is derived that

$$\begin{aligned} u'' &= 2\alpha_1(\frac{G'}{G})^3 + 3\alpha_1\lambda(\frac{G'}{G})^2 + \\ &(\alpha_1\lambda^2 + 2\alpha_1\mu)(\frac{G'}{G}) + \alpha_1\lambda\mu \end{aligned} \tag{14}$$

By substituting (11) - (14) into Eq. (8) and collecting all terms with the same power of $(\frac{G'}{G})$ together, the left-hand side of Eq. (8) is converted into another polynomial in $(\frac{G'}{G})$. Equating each coefficient of this polynomial to zero, yields a set of simultaneous algebraic equations for α_1, α_0, v, λ, μ and c as follows:

$$-\frac{1}{2}\alpha_1^3 + 2(\alpha + \beta^2)\alpha_1 = 0$$

$$\frac{3}{2}\alpha_1^2 - \frac{3}{2}\alpha_1^2\alpha_0 + 3(\alpha + \beta^2)\alpha_1\lambda = 0$$

$$-\omega^2\alpha_1 + 3\omega\alpha_1\alpha_0 - \frac{3}{2}\alpha_1\alpha_0^2 + (\alpha + \beta^2)(\alpha_1\lambda^2 + 2\alpha_1\mu) = 0$$

$$-\omega^2\alpha_0 + \frac{3}{2}\omega\alpha_0^2 - \frac{1}{2}\alpha_0^3 + (\alpha + \beta^2)\alpha_1\lambda\mu + c = 0$$

On solving the algebraic equations above by maple package we obtain

$$\alpha_1 = \pm 2\sqrt{\alpha + \beta^2}$$

If $\alpha_1 = 2\sqrt{\alpha + \beta^2}$ we have

$$\omega = \pm\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}$$

And for

$$\omega = \sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}$$

We have

$$\alpha_0 = \frac{\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\sqrt{\alpha + \beta^2} + \lambda\alpha + \lambda\beta^2}{\sqrt{\alpha + \beta^2}}$$

By using (16), expression (11) can be written as

$$u(\xi) = 2\sqrt{\alpha + \beta^2}\, i(\frac{G}{G}) + \frac{\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\sqrt{\alpha + \beta^2} + \lambda\alpha + \lambda\beta^2}{\sqrt{\alpha + \beta^2}}$$

And

$$\xi = x - \sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\, t \cdot$$

Substituting the general solutions of Eq. (10) as follows

$$\frac{G}{G} = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) - \frac{\lambda}{2}$$

Into (18) we have three types of travelling wave solutions of the Cupled Witham-Broer-Kaup equations (6) as follows:

When $\lambda^2 - 4\mu \succ 0$

$$u(\xi) = \sqrt{(\alpha + \beta^2)(\lambda^2 - 4\mu)}\left(\frac{C_1 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right) + \frac{\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\sqrt{\alpha + \beta^2} + \lambda\alpha + \lambda\beta^2}{\sqrt{\alpha + \beta^2}} - \frac{\lambda}{2}$$

Where $\xi = x - \sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\, t$. C_1 and C_2 are arbitrary constants.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda \succ 0$, $\mu = 0$, u, become

$$u(\xi) = \lambda\sqrt{\alpha + \beta^2}\, tgh\frac{1}{2}\lambda\xi\, \frac{\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha\lambda^2 - 2\beta\lambda^4}\sqrt{\alpha + \beta^2} + \lambda\alpha + \lambda\beta^2}{\sqrt{\alpha + \beta^2}} - \frac{\lambda}{2}$$

Which is the solitary wave solution of the Cupled Witham-Broer-Kaup equations.

When $\lambda^2 - 4\mu \prec 0$

$$u(\xi) = \sqrt{(\alpha + \beta^2)(\lambda^2 - 4\mu)}\left(\frac{-C_1 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right) + \frac{\sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\sqrt{\alpha + \beta^2} + \lambda\alpha + \lambda\beta^2}{\sqrt{\alpha + \beta^2}} - \frac{\lambda}{2}$$

When $\lambda^2 - 4\mu = 0$

$$u(\xi) = \frac{2\sqrt{\alpha + \beta^2}C_2}{C_1 + C_2\xi}, \quad \xi = x - \sqrt{3\lambda^2\alpha + 3\lambda^2\beta^2 - 2\alpha(\lambda^2 + 2\mu) - 2\beta(\lambda^2 + 2\mu)^2}\, t$$

Where C_1 and C_2 are arbitrary constants.

And for $\alpha_1 = -2\sqrt{\alpha + \beta^2}$ we apply this above method.

CONCLUSIONS

These equations are very difficult to be solved by traditional methods. The $(\frac{G'}{G})$-expansion method has its own advantages: direct, concise, elementary that the general solutions of the second order LODE have been well known for many researchers and effective that it can be used for many other nonlinear evolution equations.

REFERENCES

1. Parkes, E.J. 1996. B.R. Duffy, An automated tanh-function method for finding solitary wave solutions to nonlinear evolution equations, Comput. Phys. Commun. 98: 288300.
2. Fan, E.G., 2000. Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A, 277: 212-218.
3. Liu, S.K., Z.T. Fu, S.D. Liu and Q. Zhao, 2001. New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations,Phys. Lett. A, 289: 69-74.
4. Fu, Z.T., S.K. Liu, S.D. Liu and Q. Zhao, 2001. Periodic solution of Kdv Burger equation, Phys. Lett. A, 290: 72-76.
5. Wang, M.L., X.Z. Li and J.L. Zhang, 2008. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics,Phys. Lett. A, 372: 417-423.