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An Analytical Technique for Shock-Peakon and Shock-Compacton Solutions

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Abstract: This paper outlines the implementation of variational iteration method (VIM) for finding new solitary solutions for nonlinear dispersive *K (p, q)* equations. Numerical results coupled with the graphical represnation explicitly reveal the accuracy, simplicity and efficiency of the proposed algorithm.

Key words: Variational iteration method · Lagrange multiplier · Shock-peakon solution · Shock-compacton solution · *K (p, q)* equation

INTRODUCTION

The nonlinear *K(p, q)* equations [1-36] arise very frequently in various physical phenomenon and a new type of solution which is named as peakon and compacton is of utmost importance in this context. The generalized form of a nonlinear dispersive equation *K(p, q) is* given by:

$$u_t + \alpha(u^p)_x + (u^q)_{xxx} = 0. \tag{1}$$

Due to the great physical significance of (1), extensive research work has been carried out by various authors, see [1-12] and the refernces therein. The basic motivation of this paper is the implemenation of variational iteration method for finding peakon and compacton solutions of nonlinear dispersive *K(p, q)* equation. It is observed that the proposed technique (VIM) is extremely simple and is highly suitable for such problems. Numerical results coupled with the graphical represenation explicitly support our claim.

Variational Iteration Method (VIM): To illustrate the basic concept of the He's VIM, we consider the following general differential equation.

$$Lu + Nu = g(x), \tag{2}$$

Where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method [13-21, 32-40], we can construct a correction functional as follows.

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (L u_n(s) + N\tilde{u}_n(s) - g(s))\,ds, \tag{3}$$

Where λ is a Lagrange multiplier [13-21, 33-36], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [13-21, 33-36]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 consequently, the solution is given by $u = \lim_{n\to\infty} u_n$. The convergence of variational iteration method has been discussed in [20].

Solution Procedure

Shock Peakon Solution in K(2, 2) Equation: Now we consider *K(2,2)* equation:

$$ut + a(u^2)_x + (u^2)_{xxx} = 0, \tag{4}$$

To search for its solution, we can assume an initial condition in the form

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$$u(x,0) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0), \tag{5}$$

Where x_0 and c are constants. The correction functional is give by

$$u_{n+1}(x,t) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0) + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} + a(\tilde{u}_n{}^2)_x + (\tilde{u}_n{}^2)_{xxx}\right)ds.$$

Making the correctional functioanl stationary, Lagrange multiplier can be identified as $\lambda(s)$= -1, consequently

$$u_{n+1}(x,t) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0) - \int_0^t \left(\frac{\partial u_n}{\partial s} + a(u_n{}^2)_x + (u_n{}^2)_{xxx}\right)ds.$$

Following approximants are made:

$$u_0(x,t) = -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}(x+x_0), \tag{6}$$

$$u_1(x,t) = \frac{c^2}{3\sqrt{a}}t\sin\frac{\sqrt{a}}{2}(x+x_0), \tag{7}$$

$$u_2(x,t) = -\frac{c^3}{12}t^2\cos\frac{\sqrt{a}}{2}(x+x_0), \tag{8}$$

$$u_3(x,t) = -\frac{c^4\sqrt{a}}{72}t^3\sin\frac{\sqrt{a}}{2}(x+x_0), \tag{9}$$

$$\vdots$$

and so on, the rest of the components of the iteration can be deduced by Mathematical package. The solutions $u(x,t)$ are readily found in a closed form

$$u(\xi) = \begin{cases} -\frac{4c}{3a}\cos^2\frac{\sqrt{a}}{4}\xi & [\xi[\le \frac{\pi}{2} \\ 0 & \textit{otherwise} \end{cases} \tag{10}$$

Where $\xi = x + ct + x_0$. The obtained compacton solution, Eq.(14), has the same expression with that in Ref. [13]. The solution is shown in Fig. 1 with a = 1, c_0 = -1, x_0 = 0. From (10), we can find compacton solution arise as $a > 0$. Therefore, we pay more attention to what happens to the solution when $a > 0$. Assuming another initial condition as $u(x,0) = Ae^{\pm(\sqrt{-a}/2)(x+x_0)} + c_0$, we can then obtain the solution $u(x,t)$ in a closed form as.

$$u(x,t) = Ae^{-(\sqrt{-a}/2)\left|(x-(3/2)ac_0t+x_0)\right|} + c_0 \tag{11}$$

Where A,x_0 and c_0 are arbitrary constants, which are flowing peakon solutions as shown in Fig. (2) with A = 1, a=-1, c_0 = 1, x_0 = 0. Note that A in (11) is an arbitrary constant, hence we can obtain a new solitary solution called shock-peakon solution which can be written in the form.

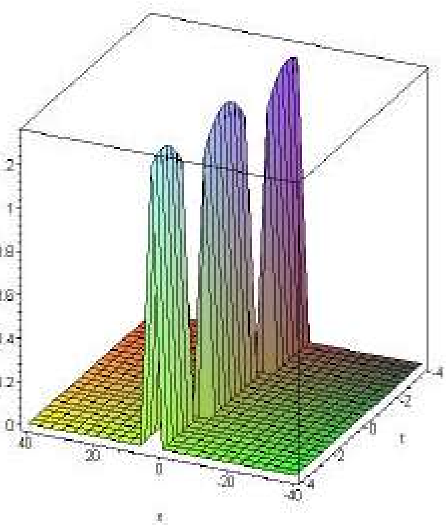

Fig. 1: Compacton solution

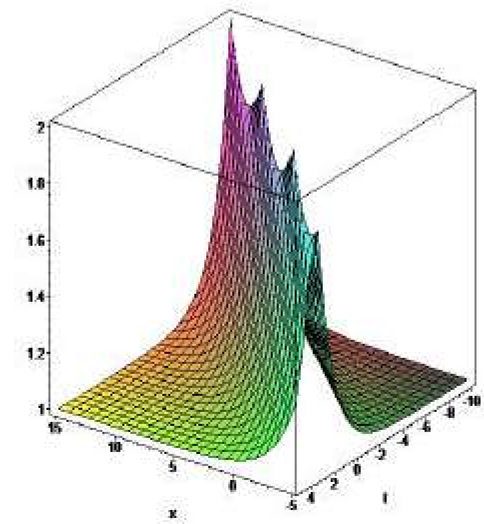

Fig. 2: Flowing peakon solution

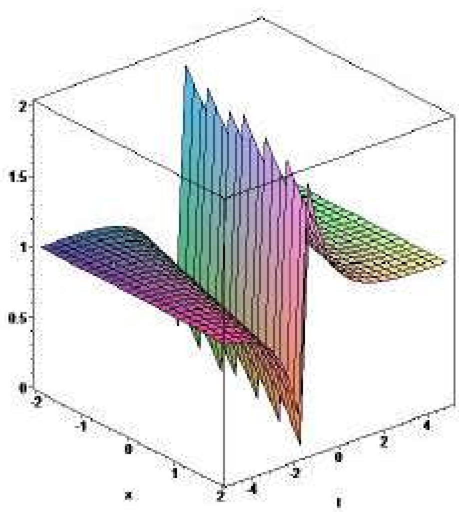

Fig. 3: Shock-peakon solution.

$$u(\xi) = Asign(\xi)e^{-(\sqrt{-a}/2)|\xi|} + c_0 \tag{12}$$

Where $\xi = x\text{-}(3/2)ac_0t + x_0$ and $sign(\xi) = \xi|\xi|$. The shock-peakon solution illustrated Fig.(3) with $A = 1$, a=-1, $c_0 = 1$, $x_0 = 0$. This is a new type of solitary waves and is a discontinuous wave. Hence it is shock wave. At the same time, it is a peakon as well. In fact, from the graphs and computation we can find that it has a discontinuous first-order derivative at $\xi = 0$. But, this solitary wave is non-local. It can be expressed by δ function. Note that the fact.

$$e^{-(\sqrt{-a}/2)|\xi|} = e^{-(\sqrt{-a}/2)sign(\xi)\xi}, \quad \frac{d}{d\xi}sign(\xi) = \delta(\xi). \tag{13}$$

Hence

$$\frac{d}{d\xi}u(\xi) = A\delta e^{(-\sqrt{-a}/2)sign(\xi)\xi} - \frac{\sqrt{-a}}{2}A\left[sign(\xi)\delta(\xi)\xi + 1\right]e^{-(\sqrt{-a}/2)sign(\xi)\xi}, \tag{14}$$

Where $\delta(\delta)$ is δ function. Assuming different initial conditions, we may obtain different exact solution in a closed form as follows:

$u(x,0)$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{a}}{4}(x+x_0)$	$-\frac{4c}{3a}\cosh^2\frac{\sqrt{-a}}{4}(x+x_0),$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{-a}}{4}(x+x_0),$
$u(x,0)$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{a}}{4}(x+ct+x_0)$	$-\frac{4c}{3a}\cosh^2\frac{\sqrt{-a}}{4}(x+ct+x_0),$	$-\frac{4c}{3a}\sin^2\frac{\sqrt{-a}}{4}(x+ct+x_0),$

Shock Peakon Solution in K(3,3) Equation: Now we consider *K(3,3)* equation:

$$u_t + a(u^3)_x + (u^3)_{xxx} \tag{15}$$

According to HPM, we readily construct the homotopy.

$$u_t + p(a(u^3)_x + (u^3)_{xxx}), \tag{16}$$

To search for its solution, we can assume an initial condition in the form

$$u(x,0) = \sqrt{-\frac{3c}{2a}\cos\frac{\sqrt{a}}{3}(x+x_0)}, \tag{17}$$

Where x_0 and c are constants. The correction functional is give by

$$u_{n+1}(x,t) = \sqrt{-\frac{3c}{2a}\cos\frac{\sqrt{a}}{3}(x+x_0)} + \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} - a(\tilde{u}_n{}^3)_x + (\tilde{u}_n{}^3)_{xxx}\right)ds.$$

Making the correctional functioanl stationary, Lagrange multiplier can be identified as $\lambda(s)$ =-1, consequently

$$u_{n+1}(x,t) = \sqrt{-\frac{3c}{2a}\cos\frac{\sqrt{a}}{3}(x+x_0)} - \int_0^t \lambda(s)\left(\frac{\partial u_n}{\partial s} - a(u_n{}^3)_x + (u_n{}^3)_{xxx}\right)ds.$$

Following approximants are made:

$$u_0(x,t) = \sqrt{-\frac{3c}{2a}\cos\frac{\sqrt{a}}{3}(x+x_0)}, \tag{18}$$

$$u_1(x,t) = -\sqrt{-\frac{c^3}{6}t\sin\frac{\sqrt{a}}{3}(x+x_0)}, \tag{19}$$

$$u_2(x,t) = -\sqrt{-\frac{c^5 a}{6}t^2\cos\frac{\sqrt{a}}{3}(x+x_0)}, \tag{20}$$

$$u_3(x,t) = \frac{1}{54}\sqrt{-\frac{c^7 a^2}{6}t^3\sin\frac{\sqrt{a}}{3}(x+x_0)}, \tag{21}$$

$$\vdots$$

and so on. The solutions $u(x,t)$ in a closed form are obtained by Mathematica package

$$u(\xi) = \begin{cases} \sqrt{-\frac{3c}{2a}\cos\frac{\sqrt{a}}{3}\xi} & [\xi[\le \frac{\pi}{2} \\ 0 & otherwise \end{cases} \tag{22}$$

Where $\xi = x + ct + x_0$. The compacton solution is shown in Fig.(4) with a=-1, c_0 = 1, x_0 = 0.

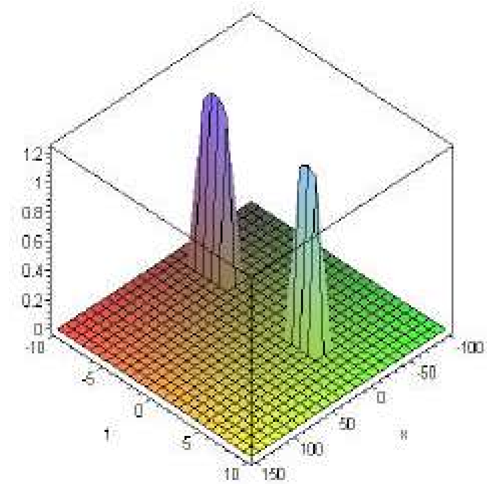

Fig. 4: Compacton solution

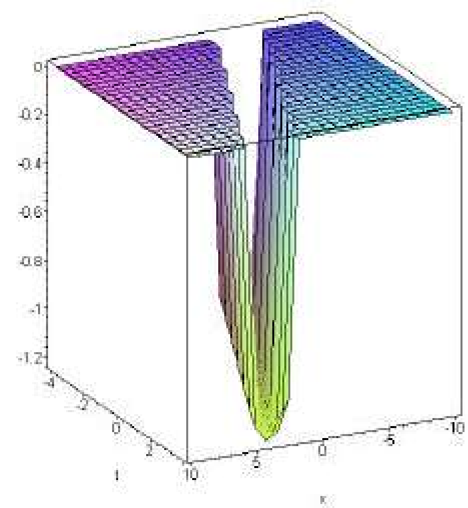

Fig. 5: Shock-compacton solution

By the same method, we can obtain the solution $u(x,t)$ in a closed form as

$$u(\xi)=\begin{cases}-\sqrt{-\dfrac{3c}{2a}}\cos\dfrac{\sqrt{a}}{3}\xi & |\xi|\le\dfrac{\pi}{2},\\ 0 & otherwise,\end{cases} \tag{23}$$

So we can obtain a new solitary solution called the shock-compacton solution as follows

$$u(\xi)=\begin{cases}\sqrt{-\dfrac{3c}{2a}}sign(\xi)\cos\dfrac{\sqrt{a}}{3}\xi & |\xi|\le\dfrac{\pi}{2},\\ 0 & otherwise,\end{cases} \tag{24}$$

Which is shown in Fig.(5) with a=-1, c_0 = 1, x_0 = 0. Form the graphs and computation we can find that it has discontinuous first-order derivative at $\xi = 0, \pm \pi/2$. Note that the fact

$$u(\xi)=\sqrt{-\frac{3c}{2a}}sign(\xi)A(\xi)\cos\frac{\sqrt{a}}{3}\xi \tag{25}$$

$$\frac{d}{d\xi}A(\xi)=\delta\left(\xi+\frac{\pi}{2}\right)-\delta\left(\xi-\frac{\pi}{2}\right)\quad\text{and}\quad\frac{d}{d\xi}sign(\xi)=\delta(\xi) \tag{26}$$

We have

$$\begin{aligned}\frac{d}{d\xi}u(\xi)&=\sqrt{-\frac{3c}{2a}}\delta(\xi)A(\xi)\cos\frac{\sqrt{a}}{3}\xi+\sqrt{-\frac{3c}{2a}}sign(\xi)\left[\delta\left(\xi+\frac{\pi}{2}\right)-\delta\left(\xi-\frac{\pi}{2}\right)\right]\cos\frac{\sqrt{a}}{3}\xi\\&\quad-\sqrt{-\frac{c}{6}}sign(\xi)A(\xi)\sin\frac{\sqrt{a}}{3}\xi,\end{aligned} \tag{27}$$

Where

$$A(\xi)=\begin{cases}1, & |\xi|\le\dfrac{\pi}{2}\\ 0 & otherwise,\end{cases} \tag{28}$$

$\delta(\delta)$ is δ function. Hence shock-compacton solution is non-local and new type of solitary wave. It has the characters of shock and compacton.

CONCLUSION

In this study, we obtain two new types of solitary wave solution: shock-peakon and shock-compacton for *K(p, q)* equation by means of the variatioanl iteration method. They are non-local shock wave solutions, having the characters of peakon and compacton. Nnumerical results clearly reveal the complete reliablity of the proposed algorithm.

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