World Applied Sciences Journal 10(12): 1490-1492, 2010
ISSN 1818-4952
© IDOSI Publications, 2010

Approximate Solution of Helmholtz Equation by Differential Transform Method

Najeeb Alam Khan, Asmat Ara and Ahmet Yildirim

Department of Mathematics, University of Karachi, Karachi-75270, Pakistan

Abstract: A variety of problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic, etc) lead to the two-dimensional Helmholtz equation. The differential transform method (DTM)provides an effective tool to numerically solve partial differential equations. In this letter, we propose numerical solution of Helmholtz partial differential equation by two dimensional differential transform method (DTM). Numerical example justify the proposed scheme.

Key words: Differential transform method (DTM) · Helmholtz equation · Analytical solution

INTRODUCTION

Mathematical modeling of many physical systems leads to partial differential equations in various fields of physics and engineering. Analytical solutions of these equation may well describe the various phenomenon in science and nature, such as vibrations, plasma physics, fluid mechanics, solid state physics, optical fiber and chemical kinetics etc. Many effective methods have been proposed for finding the analytical solutions. For example, Adomian decomposition method (ADM) [1], homotopy perturbation method (HPM) [2], variation iteration method (VIM) [3], homotopy analysis method (HAM) and differential transform method (DTM). The homotopy perturbation method (HPM) and variation iteration method (VIM) proposed by Ji-Huan He. On the other hand homotopy analysis method (HAM) is developed and proposed by Shijun Liao [4] in 1992. These methods provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions of differential equations.

The differential transform method (DTM) was first introduced by Zhou [5] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative process for obtaining the analytic Taylor series solution of ordinary or partial differential equations.

The Helmholtz equation $\Delta_2 w + \lambda w = -\phi(x,y)$ is an important equation of physics. Many problems related to steady-state oscillations (mechanical, acoustical, thermal, electromagnetic, etc) lead to the two-dimensional Helmholtz equation with $\lambda > 0$. This equation governs the mass transfer phenomena with volume chemical reaction of the first order for $\lambda < 0$. This equation is also appears in Nuclear reactors and Lamb problem in geophysics [6-8]. Any elliptic equation with constant coefficient can be reduced to Helmholtz equation.

In the present work we implement DTM for finding the approximate solution of the Helmholtz equation. We consider

$$\Delta_2 w + \lambda w = -\phi(x,y) \tag{1}$$

With the initial and boundary conditions

$$w(0,y) = f_1(y), \quad w_x(0,y) = f_2(y)$$

$$w(x,0) = f_3(x), \quad w_y(x,0) = f_4(x) \tag{2}$$

Where $f_1(y), f_2(y), f_3(x), f_4(x)$ are given functions.

The basic definitions and fundamental operation of the two dimensional DTM in [9] as follows:

Consider a function $w(x,y)$ of two variables, be analytic in the domain K and let $(x,y) = (x_0, y_0)$ in this domain. The function $w(x,y)$ is then represented by one series whose centre is located at (x_0, y_0). The differential transform of the function $w(x,y)$ is the form

Corresoponding Author: Najeeb Alam Khan, Department of Mathematics, University of Karachi, Karachi-75270, Pakistan.
E-mail: njbalam@yahoo.com.

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(x,y)}{\partial x^k\,\partial y^h}\right]_{(0,0)} \tag{3}$$

Where $w(x,y)$ is the original function and $W(k,h)$ is the transformed function.

The differential inverse transform of $W(k,h)$ is defined as

$$w\,(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)\;x^k y^h \tag{4}$$

and from equation (3) and (4) can be concluded

$$w\,(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}w(x,y)}{\partial x^k\partial y^h}\right]_{(0,0)}\;x^k y^h$$

The fundamental operations performed by two dimensional DTM are listed in Table 1.

In order to asses the advantages of DTM, we will consider the following examples.

Example 1: Consider the case for $\lambda < 0$ of Helmholtz equation

$$\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}-w(x,y)=0 \tag{5}$$

With the initial conditions

$$w(0,y)=y,\; w_x(0,y)=y+Coshy \tag{6}$$

The transformed version of equation (5) is

$$(k+1)(k+2)\,W(k+2,h)+(h+1)(h+2)W(k,h+2)-W(k,h)=0 \tag{7}$$

By applying the initial condition (6) into equation (4), the initial transformation coefficients are thus determined by

$$\sum_{h=0}^{\infty}W\,(0,h)y^h=y\;,\;\sum_{h=0}^{\infty}W\,(1,h)y^h=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+..... \tag{8}$$

Substituting (8) in equation (7), we obtained the closed form series solution as

$$w\,(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)\;x^k y^h \tag{9}$$

$$w(x,y)=x\left(1+\frac{y^2}{2!}+\frac{y^4}{4!}+.....\right)+y\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+.....\right)$$

The solution $w(x,y)$ in a closed form is

$$w(x,y)=x\,Coshy+ye^x \tag{10}$$

Example 2: Consider a special case for $\lambda > 0$ of Helmholtz equation

$$\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}+5w(x,y)=0 \tag{11}$$

with the initial conditions

$$w(0,y)=y,\; w_x(0,y)=3\,Sinh2y \tag{12}$$

The transformed version of equation (11) is

$$(k+1)(k+2)\,W(k+2,h)+(h+1)(h+2)W(k,h+2)-5W(k,h)=0 \tag{13}$$

By applying the initial condition (12) into equation (4), the initial transformation coefficients are thus determined by

$$\sum_{h=0}^{\infty}W\,(0,h)y^h=0,\quad \sum_{h=0}^{\infty}W\,(1,h)y^h=3\left(2y+\frac{(2y)^3}{3!}+\frac{(2y)^5}{5!}+.....\right) \tag{14}$$

Substituting (14) into equation (11), we obtained the series solution as

$$w(x,y)=\left(3x-\frac{(3x)^3}{3!}+\frac{(3x)^5}{5!}-....\right)\left(2y+\frac{(2y)^3}{3!}+\frac{(2y)^5}{5!}+.....\right) \tag{15}$$

The solution $w(x,y)$ in a closed form is

$$w(x,y)=Sin3x\;Sinh2y \tag{16}$$

Closing Remarks: In this letter, the DTM has been successfully applied to find the solution of Helmholtz partial differential equation and this equation appear in diverse phenomena such as elastic wave in solids including vibrating string, bars, sounds and acoustics, electromagnetic waves and nuclear reactors. Also this analysis exhibits the applicability of the DTM to solve the differential equation. This work emphasized our belief that the method is a reliable technique to handle the partial differential equations. It provides the solutions in terms of convergent series with the easily computable components in a direct way without perturbation, discretization or restrictive assumptions. The results of this method are in good agreement with those of obtained by using VIM [3] and ADM [1]. As an advantage of this method over the other, the method reduces the computational difficulties and calculations can be made simple manipulations.

REFERENCES

1. El-Sayed, M. and D. Kaya, 2004. App. Math. Comput., 150: 763-773.
2. Khan, N.A., A. Ara, S.A. Ali and A. Mahmood, 2009. Int. J. Non-Lin Sc. & Num. Simul., 10 (in press).
3. Momani, S. and S. Abuasad, 2007. Chaos Solitons & Fractals, 27: 1119-1123.
4. Liao, S.J., 1992. Ph.D Thesis, Shanhgai Jiao Tong University.
5. Zhou, J.K., 1986. Differential transformation and its application in electric circuits, Huazhong University press, Wuhan, Chaina.
6. Burden, R.L. and J.D. Faires, 1993. Numerical Analysis Boston PWS Publishing Company.
7. Gerald, C.F. and P.O. Wheatley, 1994. Applied Numerical Analysis California Addison Wisely.
8. Debnath, L., 2004. Nonlinear partial differential equations for scientist and engineers Boston Birkhauser.
9. A-H Hassan, I.H., 2008. Chaos, Solitons & Fractals 36: 53-65.

Original function	Transformed function
$w(x,y) = w_1(x,y) + w_2(x,y)$	$W(k,h) = W_1(k,h) + W_2(k,h)$
$w(x,y) = \alpha\, w_1(x,y)$	$W(k,h) = \alpha\, W_1(k,h)$
$w(x,y) = \dfrac{\partial w_1(x,y)}{\partial x}$	$W(k,h) = (k+1)\, W_1(k+1,h)$
$w(x,y) = \dfrac{\partial w_1(x,y)}{\partial y}$	$W(k,h) = (h+1)\, W_1(k,h+1)$
$w(x,y) = \dfrac{\partial^{r+s} w_1(x,y)}{\partial x^r \partial y^s}$	$W(k,h) = (k+1)(k+2)...(k+r)(h+1)(h+2)........(h+s)\, W_1(k,h+2)$