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On a System of Rational Difference Equations

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Abstract: In this paper, we have studied the stability of the difference equation system $x_{n+1}=\dfrac{x_n+y_{n-2}}{x_ny_{n-2}+1}$, $y_{n+1}=\dfrac{y_n+x_{n-2}}{y_nx_{n-2}+1}$, $n=0,1,2,...$ where $x_{-2},x_{-1},x_{-0},y_{-2},y_{-1},y_0$ are positive real numbers.

Key words: Difference equation system · Stability · Periodicity

INTRODUCTION

Our aim in this paper is to investigate the stability of the difference equation system.

$$x_{n+1}=\frac{x_n+y_{n-2}}{x_ny_{n-2}+1}\ ,\ y_{n+1}=\frac{y_n+x_{n-2}}{y_nx_{n-2}+1}\ ,\ n=0,1,2,... \tag{1}$$

Where

$$x_{-2},x_{-1},x_0,y_{-2},y_{-1},y_0\in R^+ \tag{2}$$

Some Papers Related to this Subject Are the Following: Cinar [1] has obtained a sufficient condition for the global stability of the difference equation system

$$z_{n+1}=\frac{z_nt_{n-1}+a}{z_n+t_{n-1}}\ ,\ t_{n+1}=\frac{t_nz_{n-1}+a}{t_n+z_{n-1}}\ ,\ n=0,1,2,... \tag{3}$$

Where $\alpha\in(0,\infty)$ and (z_k, t_k) for k=-1,0..

Li and Zhu [2] proved that the unique positive equilibrium of the difference equation

$$x_{n+1}=\frac{x_nx_{n-1}+a}{x_n+x_{n-1}}\ ,\ n=0,1,... \tag{4}$$

Where $\alpha\in(0,\infty)$ and x_{-1},x_0 are positive, is globally asymptotically stable.

Berenhaut, Foley and Stevic [3] has showed that the unique positive equilibrium $\bar{y}=1$ of the difference equation

$$y_n=\frac{y_{n-k}+y_{n-m}}{1+y_{n-k}y_{n-m}}\ ,\ n=0,1,... \tag{5}$$

is globally asymptotically stable.

Moreover, Amleh, Kruse and Ladas [4] proved that all positive solutions of the difference equations:

$$x_{n+1}=\frac{x_n+x_{n-1}x_{n-2}}{x_nx_{n-1}+x_{n-2}}\ ,\ x_{n+1}=\frac{x_{n-1}+x_nx_{n-2}}{x_nx_{n-1}+x_{n-2}}\ ,\ x_{n+1}=\frac{x_n+x_{n-1}x_{n-2}}{x_nx_{n-2}+x_{n-1}} \tag{6}$$

Where the initial values x_{-2},x_{-1},x_0 are positive, converge to 1 as $n\to\infty$.

Abu-Saris, Cinar and Yalcinkaya [5] have proved that the equilibrium solution of the difference equation

$$y_{n+1}=\frac{y_ny_{n-k}+a}{y_n+y_{n-k}}\ ,\ n=0,1,.... \tag{7}$$

Where k is a non-negative integer, $\alpha\in[0,\infty)$ and $x_{-k,...,}x_0$ are positive, is globally asymptotically stable.

In this paper, in a similar way to the before mentioned works, we define the equation system (1) with conditions (2) and investigate the solutions of this difference equation system. Here, we review some results [6] which will be useful in our investigation of the behavior of the solutions of the system (1).

Let I be some interval of real numbers and let f, g : $I\times I\to I$ be continuously differentiable functions. Then for all initial values $(x_k, y_k)\in I$ and $k=-2,-1,0$ the system of difference equation

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$$x_n+1 = f(x_n, y_{n-2}),\ y_{n+1} = g(y_n, x_{n-2}),\ n = 0,1,... \quad (8)$$

has a unique solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$.

Definition 1.1: A point $(\bar{x},\bar{y})$ is called an equilibrium point of the system (1) if.

$$\bar{x} = f(\bar{x},\bar{y}) \text{ and } \bar{y} = g(\bar{x},\bar{y}) \quad (9)$$

Definition 1.2: Let $(\bar{x},\bar{y})$ be an equilibrium point of the system (1).

- An equilibrium point $(\bar{x},\bar{y})$ is said to be stable if for any ε > 0 there is δ > 0 such that for every initial points (x_{-2}, y_{-2}),(x_{-1}, y_{-1}) and (x_0, y_0) for which

$$\|(x_{-2},y_{-2})-(\bar{x},\bar{y})\|+\|(x_{-1},y_{-1})-(\bar{x},\bar{y})\|+\|(x_0,y_0)-(\bar{x},\bar{y})\|<\delta$$

the iterates (x_n, y_n) of (x_{-2}, y_{-2}),(x_{-1}, y_{-1}) and (x_0, y_0) satisfy

$$\|(x_n,y_n)-(\bar{x},\bar{y})\|<\varepsilon \quad \text{, for all } n>0 \quad (10)$$

An equilibrium point $(\bar{x},\bar{y})$ is said to be unstable if it is not stable. (By ||.|| we denote the Euclidean norm in R^2 given by $\|(x_n,y_n)\|=\sqrt{x^2+y^2}$).

- An equilibrium point $(\bar{x},\bar{y})$ is said to be asymptotically stable if there exists $r > 0$ such that $(x_n, y_n) \to \infty$ as $n \to \infty$ for all (x_{-2}, y_{-2}), (x_{-1}, y_{-1}) and (x_0, y_0) that satisfy.

$$\|(x_{-2},y_{-2})-(\bar{x},\bar{y})\|+\|(x_{-1},y_{-1})-(\bar{x},\bar{y})\|+\|(x_0,y_0)-(\bar{x},\bar{y})\|<r \quad (11)$$

Definition 1.3: Let $(\bar{x},\bar{y})$ be an equilibrium point of a map $F = (f, g)$, where f and g are continuously differentiable functions at $(\bar{x},\bar{y})$. The Jacobian Matrix of F at $(\bar{x},\bar{y})$ is the matrix.

$$J_F(\bar{x},\bar{y})=\begin{bmatrix}\frac{\partial f}{\partial x}(\bar{x},\bar{y}) & \frac{\partial f}{\partial y}(\bar{x},\bar{y})\\ \frac{\partial g}{\partial x}(\bar{x},\bar{y}) & \frac{\partial g}{\partial y}(\bar{x},\bar{y})\end{bmatrix} \quad (12)$$

The linear map

$$J_F(p,q)(\bar{x},\bar{y})=\begin{bmatrix}\frac{\partial f}{\partial x}(\bar{x},\bar{y})x+\frac{\partial f}{\partial y}(\bar{x},\bar{y})y\\ \frac{\partial g}{\partial x}(\bar{x},\bar{y})x+\frac{\partial g}{\partial y}(\bar{x},\bar{y})y\end{bmatrix} \quad (13)$$

is called the linearization of the map F at $(\bar{x},\bar{y})$.

Theorem 1.4: Let $F = (f, g)$ be a continuously differentiable function defined on an open set I in R^2 and let $(\bar{x},\bar{y})$ be an equilibrium point of the map $F = (f, g)$.

- If all the eigenvalues of the Jacobian matrix $J_F(\bar{x},\bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x},\bar{y})$ is asymptotically stable.
- If at least one of the eigenvalues of the Jacobian matrix $J_F(\bar{x},\bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x},\bar{y})$ is unstable.
- An equilibrium point $(\bar{x},\bar{y})$ of the map $F = (f, g)$ is locally asymptotically stable if and only if every solution of the characteristic equation.

$$\lambda^2 - trJ_F(\bar{x},\bar{y})\lambda + det\,J_F(\bar{x},\bar{y}) = 0 \quad (14)$$

lies inside the unit circle, that is, if and only if

$$|trJ_F(\bar{x},\bar{y})| < 1 + det\,J_F(\bar{x},\bar{y}) < 2 \quad (15)$$

Definition 1.5: Let $(\bar{x},\bar{y})$ be a positive equilibrium point of the system (1).

A string of consecutive terms $\{x_s,...,x_m\}$ (resp. $\{x_s,...,x_m\}$), $s \geq -1$, m is said to be a positive semicycle if $x_i \geq \bar{x}$ (resp. $y_i \geq \bar{y}$), $i \in$, $\{s,...,m\}$ (resp. $x_{s-1} < \bar{x}$ ($y_{s-1} < \bar{y}$) and $x_{m+1} < \bar{x}$ (resp. $y_{m+1} < \bar{y}$).

A string of consecutive terms $\{x_s,...,x_m\}$ (resp. $\{y_s,...,y_m\}$), $\{y_s,...,y_m\}$ $s \geq -1$, $m < \infty$ is said to be a negative semicycle if $x_i < \bar{x}$ (resp. $y_i < \bar{y}$), $I \in$, $\{y_s,...,y_m\}$ $x_{I-i} \geq \bar{x}$ (resp. $y_{s-1} \geq \bar{y}$) and $x_{m+1} \geq \bar{x}$ (resp. $y_{m+1} \geq \bar{y}$).

A string of consecutive terms $\{y_s,...,y_m\}$(resp. $\{y_s,...,y_m\}$), s ≥ −1, $m < \infty$ is said to be a positive (resp. negative) semicycle if $\{y_s,...,y_m\}$, $\{y_s,...,y_m\}$ are positive (resp. negative) semicycles.

Finally, a string of consecutive terms $\{(x_s, y_s),...,(x_m, y_m)\}$ is said to be a semicycle positive (resp. negative) with respect to x_n and negative (resp. positive) with respect to y_n if $\{y_s,...,y_m\}$ is a positive (resp. negative) semicycle and $\{y_s,...,y_m\}$ is a negative (resp. positive) semicycle.

A solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of the system (1) is called non-oscillatory about $(\bar{x}, \bar{y})$, or simply non-oscillatory, if there exists $N \geq -2$ such that either $x_n \geq \bar{x}$ and $y_n \geq \bar{y}$ for all $n \geq N$ or $x_n < \bar{x}$ and $y_n < \bar{y}$ for all $n \geq N$. Otherwise, the solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ is called oscillatory about $(\bar{x}, \bar{y})$, or simply oscillatory.

Main Results: The equilibrium points $(\bar{x}, \bar{y})$ of the system (1) are the solutions of the system.

$$\bar{x} = \frac{\bar{x} + \bar{y}}{1 + \bar{x}\bar{y}} \quad , \quad \bar{y} = \frac{\bar{y} + \bar{x}}{1 + \bar{y}\bar{x}} \tag{16}$$

So $(\bar{x}, \bar{y})$ = (1, 1) is the unique positive equilibrium and $(\bar{x}, \bar{y}) = (0,0)$ is the zero equilibrium. The characteristic equations of the system (1) at (1,1) and (0,0) are respectively

$$\lambda^2 = 0 \quad \text{and} \quad \lambda^2 - 2\lambda = 0 \tag{17}$$

Theorem 2.1: Let (1,1) and (0,0) be the equilibrium points of the system (1).

- The unique positive equilibrium (1,1) is locally asymptotically stable.
- The zero equilibrium (0,0) is unstable.

Proof (1): According to (17), the eigenvalues at the unique positive equilibrium (1,1) are $\lambda_1 = \lambda_2$. Therefore, the system (1) is locally asymptotically stable at (1,1).
(2) According to (17), the eigenvalues at the zero equilibrium (1,1) are $\lambda_1 = 0$ and $\lambda_2 = 2$. Thus, the system (1) is unstable at (0,0).

Lemma 2.2: A positive solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of the system (1) is eventually equal to (1,1) if and only if

$$(x_{-1} - 1)(x_0 - 1)(y_{-2} - 1)(y_0 - 1) = 0 \tag{18}$$

Proof: Let x_{-2} −1. Then from (1),

$$y_1 = \frac{y_0 + x_{-2}}{1 + y_0 x_{-2}} = \frac{y_0 + 1}{y_0 + 1} = 1,\ y_2 = \frac{1 + x_{-1}}{x_{-1} + 1} = 1,\ y_3 = \frac{1 + x_0}{x_0 + 1} = 1$$

and

$$x_1 = \frac{x_0 + y_{-2}}{1 + x_0 y_{-2}},\ x_2 = \frac{x_1 + y_{-1}}{1 + x_1 y_{-1}},\ x_3 = \frac{x_2 + y_0}{1 + x_2 y_0},$$

$$x_4 = \frac{x_3 + y_1}{1 + x_3 y_1} = \frac{x_3 + 1}{1 + x_3} = 1,\ x_5 = \frac{x_4 + 1}{1 + x_4} = 1,\ x_6 = \frac{x_5 + 1}{1 + x_5} = 1$$

From (1), $(x_i, y_i) = (1,1)$, for $(i = 4,5,...)$. Similarly, for $x_0 = 1$, $y_{-2} = 1$ and $y_0 = 1$ we get $(x_i, y_i) = (1,1)$, for $(i = 4,5,...)$. Conversely, for $t \in \{-2,0\}$ assume that

$$x_t \neq 1 \text{ or } y_t \neq 1 \tag{20}$$

Then we must show that

$$x_n \neq 1 \text{ or } y_n \neq 1 \text{ for } n \geq 1 \tag{21}$$

Consider that for some $N \geq 1$,

For $-1 \leq n \leq N-1$, $x_N = y_N = 1$ and $x_n \neq 1$, $y_n \neq 1$ (22)

We can easily say that

$$1 = x_N = \frac{x_{N-1} + y_{N-3}}{1 + x_{N-1} y_{N-3}} \Rightarrow (x_{N-1} - 1)(y_{N-3} - 1) = 0 \tag{23}$$

From (23), $x_{N-1} = 1$ or $y_{N-3} = 1$. This contradicts equation (22).

Lemma 2.3: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1) which is not eventually equal to (1,1). Then the following statements are true:

(i) $(x_{n+1} - x_n)(x_n - 1) < 0$ and $(y_{n+1} - y_n)(y_n - 1) < 0$, for $n \geq 0$
(ii) $(x_{n+1} - x_n)(x_n - 1)(1 - y_{n-2}) > 0$ and $(y_{n+1} - y_n)(y_n - 1)(1 - x_{n-2}) > 0$, for $n \geq 0$

Proof: From the system (1),

$$x_{n+1} - x_n = \frac{y_{n-2}(1 - x_n)(1 + x_n)}{1 + x_n y_{n-2}} \text{ and } y_{n+1} - y_n = \frac{x_{n-2}(1 - y_n)(1 + y_n)}{1 + y_n x_{n-2}} \tag{24}$$

and

$$x_{n+1}-1=\frac{(x_n-1)(1-y_{n-2})}{1+x_ny_{n-2}} \text{ and } y_{n+1}-1=\frac{(y_n-1)(1-x_{n-2})}{1+y_nx_{n-2}} \quad (25)$$

for $n = 0,1,...$ from which inequalities in (i) and (ii) follow.

Lemma 2.4: If $x_k, y_k < 1$ for $k = -2,-1,0$ then (x_n, y_n) is a negative semicycle of the system (1) with an infinite number of terms and it monotonically tends to the unique positive equilibrium $(\bar{x},\bar{y})=(1,1)$.

Proof: If $x_k, y_k < 1$ for $k = -2,-1,0$, then from Lemma 2.3.[ii] and [i],

$$0 < x_0 < x_1 < ... < x_n < 1 \text{ and } 0 < y_0 < x_1 < ... < y_n < 1 \quad (26)$$

Clearly,(x_n, y_n) is a negative semicycle of the system (1) with an infinite number of terms. Furthermore, we know that (x_n, y_n) is strictly increasing for $n \geq 0$. So the limits

$$\lim_{n\to\infty} x_n = \ell_1 \text{ and } \lim_{n\to\infty} y_n = \ell_2 \quad (27)$$

exists and are finite. Taking the limits on both sides of the system (1), we have

$$\ell_1 = \frac{\ell_1+\ell_2}{1+\ell_1\ell_2} \text{ and } \ell_2 = \frac{\ell_2+\ell_1}{1+\ell_2\ell_1} \quad (28)$$

thus $\ell_1 = \ell_2 = 1$.

Lemma 2.5: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1), and consider the following cases:

1: $x_{-2} > 1$ and $x_{-1}, x_0, y_{-2}, y_{-1}, y_0 < 1$
2: $x_{-1} > 1$ and $x_{-2}, x_0, y_{-2}, y_{-1}, y_0 < 1$
3: $y_0 > 1$ and $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} < 1$
4: $x_{-1}, y_0 > 1$and $x_{-2}, x_0, y_{-2}, y_{-1} < 1$
5: $y_{-2}, y_{-1} > 1$ and $x_{-2}, x_{-1}, x_0, y_0 < 1$
6: $y_{-2}, y_0 > 1$ and $x_{-2}, x_{-1}, x_0, y_{-2} < 1$
7: $y_{-2}, y_{-1}, y_0 > 1$ and $x_{-2}, x_{-1}, x_0 < 1$
8: $y_{-2}, y_{-1}, x_{-1} > 1$ and $x_{-2}, x_0, y_0 < 1$
9: $x_{-2}, x_0, y_{-2} > 1$ and $x_{-1}, y_{-1}, y_0 < 1$
10: $x_{-2}, y_{-1}, y_0 > 1$ and $x_{-1}, x_0, y_{-2} < 1$
11: $x_0, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-2}, x_{-1} < 1$
12: $x_{-1}, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-2}, x_0 < 1$
13: $x_{-2}, x_0\, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-1} < 1$

If one of the above cases occurs, then

- Every positive semicycle associated with $\{x_n\}$ of the system (1) consists of one term and negative semicycle associated with $\{x_n\}$ of the system (1) consists of five, three or one terms;
- Every positive semicycle associated with $\{y_n\}$ of the system (1) consists of six or two terms and negative semicycle associated with $\{y_n\}$ of the system (1) consists of four or two terms;
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $1^+,5^-,1^+,1^-,1^+,1^-,1^+,3^-$;
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $6^+,2^-,2^+,4^-$.

Proof: Consider x_{-2} .1 and $x_{-1}, x_0, y_{-2}, y_{-1}, y_0 <1$, then in view of inequality (ii) of Lemma 2.3. we have:
$x_{-1}<$, $y_1>1$; $x_2<1$, $y_2>1$; $x_3<1$, $y_3>1$; $x_4>1$, $y_4>1$; $x_5<1$, $y_5>1$; $x_6>1$,$y_6>1$; $x_7<1$, $y_7<1$; $x_8>1$, $y_8>1$; $x_9<1$, $y_9>1$; $x_{10}<1$, $y_{10}>1$; $x_{11}<1$, $y_{11}<1$; $x_{12}>1$, $y_{12}<1$; $x_{13}<1$, $y_{13}>1$; $x_{14}<1$, $y_{14}<1$ which imply that a positive semicycle associated with $\{x_n\}$ of length one is followed by a negative semicycle of length five, three or one which in turn is followed by a positive semicycle length one. Similarly, a positive semicycle associated with $\{y_n\}$ of length six is followed by a negative semicycle of length two, the negative semicycle of length two is followed by a positive semicycle of length two, the positive semicycle of length two is followed by a negative semicycle of length four.

The other cases can be easily shown. We omit the proofs of the following three results since they can easily be obtained in a similar way to the proof of Lemma 2.5.

Lemma 2.6: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1), and consider the following cases:

14: $x_{-2}, y_{-1} > 1$ and $x_{-1}, x_0, y_{-2}, y_0 < 1$
15: $y_{-2}, y_0 > 1$ and $x_{-2}, x_{-1}, x_0, y_{-1} < 1$
16: $x_{-2}, x_{-1}, y_{-2} > 1$ and $x_0, y_{-1}, y_0 < 1$
17: $x_{-1}, x_0, y_{-1} > 1$and $x_{-2}, y_{-2}, y_0 < 1$
18: $x_0, y_{-1}, y_0 > 1$ and $x_{-2}, x_{-1}, y_{-2} < 1$
19: $x_{-2}, x_0, y_0 > 1$ and $x_{-1}, y_{-2}, y_{-1} < 1$
20: $x_{-1}, y_{-2}, y_0 > 1$ and $x_{-2}, x_0, y_{-1} < 1$
21: $x_{-1}, x_0, y_{-2}, y_{-1} > 1$ and $x_{-2}, y_0 < 1$
22: $x_{-2}, x_0, y_{-2}, y_0 > 1$ and $x_{-1}, y_{-2} < 1$
23: $x_{-2}, x_0, y_{-2}, y_{-1} > 1$ and $x_{-1}, y_0 < 1$
24: $x_{-2}, x_{-1}, x_0\, y_{-2} > 1$ and $y_{-1}, y_0 < 1$
25: $x_{-1}, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-2}, x_0 < 1$
26: $x_{-2}, x_{-1}\, y_{-2}, y_{-1}\, y_0 > 1$ and $x_0 < 1$

If one of the above cases occurs, then

- Every positive semicycle associated with $\{x_n\}$ of the system (1) consists of three, two or one terms and negative semicycle associated with $\{x_n\}$ of the system (1) consists of three or one terms;
- Every positive semicycle associated with $\{y_n\}$ of the system (1) consists of four, two or one terms and negative semicycle associated with $\{y_n\}$ of the system (1) consists of two or one terms;
- The positive and negative semicycles associated with $\{x_n\}$ one of the form is $3^+,3^-,2^+,1^-,1^+,1^-,2^+,1^-$;
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $4^+,1^-,2^+,2^-,1^+,2^-,1^+,1^-$.

Lemma 2.7: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1), and consider the following cases:

27: $x_{-2}, x_0 > 1$ and $x_{-1}, y_{-2}, y_{-1}, y_0 < 1$
28: $x_{-1}, y_{-2} > 1$ and $x_{-2}, x_0, y_{-1}, y_0 < 1$
29: $x_{-2}, x_{-1}, y_0 > 1$ and $x_0, y_{-2}, y_{-1} < 1$
30: $x_{-1}, x_0, y_0 > 1$and $x_{-2}, y_{-2}, y_{-1} < 1$
31: $x_0, y_{-2}, y_{-1} > 1$ and $x_{-2}, x_{-1}, y_0 < 1$
32: $x_0, y_{-2}, y_0 > 1$ and $x_{-2}, x_{-1}, y_{-1} < 1$
33: $x_{-2}, y_{-2},y_{-1} > 1$ and $x_{-1}, x_0, y_0 < 1$
34: $x_{-2}, x_0, y_{-1} > 1$ and $x_{-1}, y_{-2}, y_0 < 1$
35: $x_{-1}, y_{-1}, y_0 > 1$ and $x_{-2}, x_0, y_{-2} < 1$
36: $x_{-1}, x_0, y_{-2}, y_0 > 1$ and $x_{-2}, y_{-1} < 1$
37: $x_{-2}, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-1}, x_0 < 1$
38: $x_{-2}, x_{-1}, y_{-1}\, y_0 > 1$ and $x_0, y_{-2} < 1$
39: $x_{-2}, x_{-1}\, y_{-2}, y_0 > 1$ and $x_0, y_{-1} < 1$
40: $x_{-2}, x_{-1}, x_0\, y_{-1}, y_0 > 1$ and $y_{-2} < 1$
41: $x_{-2}, x_{-1}\, x_0, y_{-2}\, y_{-1} > 1$ and $y_0 < 1$

If one of the above cases occurs, then

- Every positive semicycle associated with $\{x_n\}$ of the system (1) consists of four, two or one terms and negative semicycle associated with $\{x_n\}$ of the system (1) consists of two or one terms;
- Every positive semicycle associated with $\{y_n\}$ of the system (1) consists of three, two or one terms and negative semicycle associated with $\{y_n\}$ of the system (1) consists of three or one terms;
- The positive and negative semicycles associated with $\{x_n\}$ one of the form is $4^+,1^-,2^+,2^-,1^+,{}^2-,1^+,1^-$;
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $3^+,3^-,2^+,1^-,1^+,1^-,2^+,1^-$.

Lemma 2.8: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1), and consider the following cases:

42: $x_{-2}, y_{-2}, > 1$ and $x_{-1}, x_0, y_{-1}, y_0 < 1$
43: $x_{-1}, y_{-1} > 1$ and $x_{-2}, x_0, y_{-2}\, y_0 < 1$
44: $x_0, y_0 > 1$ and $x_{-2}, x_{-1}, y_{-2}\, y_{-1} < 1$
45: $x_{-1}, x_0, y_{-1}, y_0 > 1$ and $x_{-2}, y_{-2} < 1$
46: $x_{-2}, x_0, y_{-2}\, y_0 > 1$ and $x_{-1}, x_{-1} < 1$
47: $x_{-2}, x_{-1}, y_{-2}\, y_{-1} > 1$ and $x_0, y_0 < 1$
48: $x_{-2}, x_{-1}\, x_0, y_{-2}, y_{-}, y_0 > 1$

If one of the above cases occurs, then

- Every positive semicycle associated with $\{x_n\}$ of the system (1) consists of three or one terms and negative semicycle associated with $\{x_n\}$ of the system (1) consists of two or one terms;
- Every positive semicycle associated with $\{y_n\}$ of the system (1) consists of three or one terms and negative semicycle associated with $\{y_n\}$ of the system (1) consists of two or one terms;
- The positive and negative semicycles associated with $\{x_n\}$ one of the form is $3^+,1^-,1^+,2^-,3^+,1^-,1^+,2^-$;
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $3^+,1^-,1^+,2^-,3^+,1^-,1^+,2^-$.

Lemma 2.9: Let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be a positive solution of the system (1), and consider the following cases:

49: $x_0 > 1$ and $x_{-2}, x_{-1}, y_{-2}, y_{-1}, y_0, < 1$
50: $y_{-1}, > 1$ and $x_{-2}, x_{-1}, x_0, y_{-2}, y_0 < 1$
51: $y_{-2} > 1$ and $x_{-2}, x_{-1}, x_0, y_{-1}, y_0 < 1$
52: $x_{-2}, x_{-1} > 1$ and $x_0, y_{-2}, y_{-1}, y_0 < 1$
53: $x_{-1}, y_{-2}, y_{-1} > 1$ and $x_{-2}, x_{-1}, y_0 < 1$
54: $x_0, y_{-2}, y_0 > 1$ and $x_{-2}, x_{-1}, y_{-1} < 1$
55: $x_{-2}, y_{-2},y_{-1} > 1$ and $x_{-1}, x_0, y_0 < 1$
56: $x_{-2}, x_0, y_{-1} > 1$ and $x_{-1}, y_{-2}, y_0 < 1$
57: $x_{-1}, y_{-1}, y_0 > 1$ and $x_{-2}, x_0, y_{-2} < 1$
58: $x_{-1}, x_0, y_{-2}, y_0 > 1$ and $x_{-2}, y_{-1} < 1$
59: $x_{-2}, y_{-2}, y_{-1}\, y_0 > 1$ and $x_{-1}, x_0 < 1$
60: $x_{-2}, x_{-1}, y_{-1}\, y_0 > 1$ and $x_0, y_{-2} < 1$
61: $x_{-2}, x_{-1}\, y_{-2}, y_0 > 1$ and $x_0, y_{-1} < 1$
62: $x_{-2}, x_{-1}, x_0\, y_{-1}, y_0 > 1$ and $y_{-2} < 1$
63: $x_{-2}, x_{-1}\, x_0, y_{-2}\, y_{-1} > 1$ and $y_0 < 1$

If one of the above cases occurs, then

- Every positive semicycle associated with $\{x_n\}$ of the system (1) consists of six or two terms and negative semicycle associated with $\{x_n\}$ of the system (1) consists of four or two terms;
- Every positive semicycle associated with $\{y_n\}$ of the system (1) consists of one term and negative semicycle associated with $\{y_n\}$ of the system (1) consists of five, three or one terms;
- The positive and negative semicycles associated with $\{x_n\}$ one of the form is $6^+,2^-,2^+,4^-$
- The positive and negative semicycles associated with $\{y_n\}$ one of the form is $1^+,3^-,1^+,5^-,1^+,1^-,1^+,1^-$.

Theorem 2.10: The unique positive equilibrium $(\bar{x}, \bar{y}) = (1,1)$ of the system (1) is globally asymptotically stable.

Proof: From theorem 2.1. we know that the unique positive equilibrium $(\bar{x}, \bar{y}) = (1,1)$ of the system (1) is locally asymptotically stable. So we must show that every positive solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of the system (1) converges to $(\bar{x},\bar{y}) = (1,1)$ as $n\to\infty$. Namely, we want to prove.

$$\lim_{n\to\infty} x_n = \bar{x} = 1 \quad , \quad \lim_{n\to\infty} y_n = \bar{y} = 1 \tag{29}$$

If the solution $\{(x_n, y_n)\}_{n=-2}^{\infty}$ of the system (1) is non-oscillatory about the unique positive equilibrium point $(\bar{x},\bar{y}) = (1,1)$, then according to Lemma 2.2. and Lemma 2.4. we know that the solution is either eventually equal to (1,1) or an eventually positive one that has an infinite number of terms and monotonically tends to the unique positive equilibrium point $(\bar{x},\bar{y}) = (1,1)$ of the system (1). Therefore, equation (29) holds. So we have to show that equation (29) holds for strictly oscillatory solutions. For this, let $\{(x_n, y_n)\}_{n=-2}^{\infty}$ be strictly oscillatory about $(\bar{x},\bar{y}) = (1,1)$ of the system (1). According to Lemma 2.3.(i) and Lemma 2.9., the $\{x_n\}$ solution of the system (1) has the positive and negative semicycles of the form $6^+,2^-,2^+,4^-$. Also, $\{y_n\}$ solution of the system (1) has the positive and negative semicycles of the form $1^+,3^-,1^+,5^-,1^+,1^-,1^+,1^-$. So we have the following sequences:

$$\{x_{p+14n}, x_{p+14n+1}, x_{p+14n+2}, x_{p+14n+3}, x_{p+14n+4}, x_{p+14n+5}\}^+$$
$$\{x_{p+14n+6}, x_{p+14n+7}\}^-, \{x_{p+14n+8}, x_{p+14n+9}\}^+$$
$$\{x_{p+14n+10}, x_{p+14n+11}, x_{p+14n+12}, x_{p+14n+13}\}^-, \{y_{p+14n}\}^+$$
$$,\{y_{p+14n+1}, y_{p+14n+2}, y_{p+14n+3}\}^-, \{y_{p+14n+4}\}^+$$
$$\{y_{p+14n+5}, y_{p+14n+6}, y_{p+14n+7}, y_{p+14n+8}, y_{p+14n+9}\}^-$$
$$,\{y_{p+14n+10}\}^+, \{y_{p+14n+11}\}^-, \{y_{p+14n+12}\}^+ \{y_{p+14n+13}\}^-$$

We now have the following assertions:

(i)

$$x_{p+14n} > x_{p+14n+1} > x_{p+14n+2} > x_{p+14n+3} > x_{p+14n+4}$$
$$> x_{p+14n+5}; x_{p+14n+7} > x_{p+14n+6}\ x_{p+14n+8} > x_{p+14n+9}; x_{p+14n+13}$$
$$> x_{p+14n+12} > x_{p+14n+11} > x_{p+14n+10} \text{ and}$$
$$y_{p+14n+3} > y_{p+14n+2} > y_{p+14n+1}; y_{p+14n+9} > y_{p+14n+8}$$
$$> y_{p+14n+7} > y_{p+14n+6} > y_{p+14n+5}$$

(ii)

$$x_{p+14n+5}x_{p+14n+6} > 1; x_{p+14n+7}x_{p+14n+8} < 1;$$
$$x_{p+14n+9}x_{p+14n+10} > 1; x_{p+14n+13}x_{p+14n+14} < 1 \text{ and}$$
$$y_{p+14n}y_{p+14n+1} > 1; y_{p+14n+3}y_{p+14n+4} < 1;$$
$$y_{p+14n+4}y_{p+14n+5} > 1; y_{p+14n+9}y_{p+14n+10} < 1;$$
$$y_{p+14n+10}y_{p+14n+11} > 1; y_{p+14n+11}y_{p+14n+12} < 1;$$
$$y_{p+14n+12}y_{p+14n+13} > 1; y_{p+14n+13}y_{p+14n+14} < 1$$

inequality (I) can easily be seen from Lemma 2.3.(i). for $n = 0,1,...$

$$x_{p+14n+6} = \frac{x_{p+14n+5} + y_{p+14n+3}}{1 + x_{p+14n+5}y_{p+14n+3}} > \frac{x_{p+14n+5} + y_{p+14n+3}}{x_{p+14n+5}(1 + y_{p+14n+3})} > \frac{1}{x_{p+14n+5}}$$

$$x_{p+14n+8} = \frac{x_{p+14n+7} + y_{p+14n+5}}{1 + x_{p+14n+7}y_{p+14n+5}} < \frac{x_{p+14n+7} + y_{p+14n+5}}{x_{p+14n+7}(1 + y_{p+14n+5})} < \frac{1}{x_{p+14n+7}}$$

$x_{p+14n+9}x_{p+14n+10} > 1$ and $x_{p+14n+13}x_{p+14n+14} < 1$ can easily be shown.

$$y_{p+14n+1} = \frac{y_{p+14n} + x_{p+14n-2}}{1 + y_{p+14n}x_{p+14n-2}} > \frac{y_{p+14n} + x_{p+14n-2}}{y_{p+14n}(1 + x_{p+14n-2})} > \frac{1}{y_{p+14n}}$$

$$y_{p+14n+4} = \frac{y_{p+14n+3} + x_{p+14n+1}}{1 + y_{p+14n+3}x_{p+14n+1}} < \frac{y_{p+14n+3} + x_{p+14n+1}}{y_{p+14n+3}(1 + x_{p+14n+1})} < \frac{1}{y_{p+14n+3}}$$

$$y_{p+14n+4}y_{p+14n+5} > 1; y_{p+14n+9}y_{p+14n+10} < 1$$

$$y_{p+14n+10}y_{p+14n+11} > 1; y_{p+14n+11}y_{p+14n+12} < 1;$$
$$y_{p+14n+12}y_{p+14n+13} > 1; y_{p+14n+13}y_{p+14n+14} < 1$$

can easily be shown. From inequality (i) and (ii),

$$x_{p+14n+14} < \frac{1}{x_{p+14n+13}} < \frac{1}{x_{p+14n+12}} < \frac{1}{x_{p+14n+11}} < \frac{1}{x_{p+14n+10}} < x_{p+14n+9} < x_{p+14n+8} < \frac{1}{x_{p+14n+7}} < \frac{1}{x_{p+14n+6}} < x_{p+14n+5} < x_{p+14n+4} < x_{p+14n+3} < x_{p+14n+2} < x_{p+14n+1} < x_{p+14n} \quad (30)$$

$$y_{p+14n+14} < \frac{1}{y_{p+14n+13}} < y_{p+14n+12} < \frac{1}{y_{p+14n+11}} < y_{p+14n+10} < \frac{1}{y_{p+14n+9}} < \frac{1}{y_{p+14n+8}} < \frac{1}{y_{p+14n+7}} < \frac{1}{y_{p+14n+6}} < \frac{1}{y_{p+14n+5}} < y_{p+14n+4} < \frac{1}{y_{p+14n+3}} < \frac{1}{y_{p+14n+2}} < \frac{1}{y_{p+14n+1}} < y_{p+14n} \quad (31)$$

From equation (30) and (31), we can see that $\{x_{p+14n+14}\}_{n=0}^{\infty}$ and $\{y_{p+14n+14}\}_{n=0}^{\infty}$ are decreasing with lower bound 1. So the limits

$$\lim_{n\to\infty} x_{p+14n+14} = L_1 \text{ and } \lim_{n\to\infty} y_{p+14n+14} = L_2 \quad (32)$$

exist and are finite. From equation (30) and (31), we obtain

$$\lim_{n\to\infty} x_{p+14n+9} = \lim_{n\to\infty} x_{p+14n+8} = \lim_{n\to\infty} x_{p+14n+5} = \lim_{n\to\infty} x_{p+14n+4} = \lim_{n\to\infty} x_{p+14n+3} = \lim_{n\to\infty} x_{p+14n+2} = \lim_{n\to\infty} x_{p+14n+1} = \lim_{n\to\infty} x_{p+14n} = L_1$$

$$\lim_{n\to\infty} x_{p+14n+13} = \lim_{n\to\infty} x_{p+14n+12} = \lim_{n\to\infty} x_{p+14n+11} = \lim_{n\to\infty} x_{p+14n+10} = \lim_{n\to\infty} x_{p+14n+7} = \lim_{n\to\infty} x_{p+14n+6} = \frac{1}{L_1}$$

$$\lim_{n\to\infty} y_{p+14n+12} = \lim_{n\to\infty} y_{p+14n+10} = \lim_{n\to\infty} y_{p+14n+4} = \lim_{n\to\infty} y_{p+14n} = L_2$$

$$\lim_{n\to\infty} y_{p+14n+13} = \lim_{n\to\infty} y_{p+14n+11} = \lim_{n\to\infty} y_{p+14n+9} = \lim_{n\to\infty} y_{p+14n+8} = \lim_{n\to\infty} y_{p+14n+7} = \lim_{n\to\infty} y_{p+14n+6} = \lim_{n\to\infty} y_{p+14n+5} = \lim_{n\to\infty} y_{p+14n+3} = \lim_{n\to\infty} y_{p+14n+2} = \lim_{n\to\infty} y_{p+14n+1} = \frac{1}{L_2}$$

It suffices to verify that $L_1 = L_2$. For this,

$$x_{p+14n+14} = \frac{x_{p+14n+13} + y_{p+14n+11}}{1 + x_{p+14n+13} y_{p+14n+11}}; y_{p+14n+14} = \frac{y_{p+14n+13} + x_{p+14n+11}}{1 + y_{p+14n+13} x_{p+14n+11}} \quad (33)$$

If we take the limits on both sides of the equation (33), we obtain

$$L_1 = \frac{\frac{1}{L_1} + \frac{1}{L_2}}{1 + \frac{1}{L_1}\frac{1}{L_2}} \quad ; \quad L_2 = \frac{\frac{1}{L_2} + \frac{1}{L_1}}{1 + \frac{1}{L_2}\frac{1}{L_1}} \quad (34)$$

Which imply that $L_1 = L_2$. So we have shown that

$$\lim_{n\to\infty} x_{p+14n+k} = \lim_{n\to\infty} y_{p+14n+k} = 1 \quad \text{for} k \in \{0, ..., 14\} \quad (35)$$

Similarly, from Lemma 2.3.(i) and Lemma 2.5., Lemma 2.6., Lemma 2.7. and Lemma 2.8. one can see that equation (35) holds. Therefore, the proof is completed.

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