World Applied Sciences Journal 10(11): 1374-1378, 2010
ISSN 1818-4952

On Mannheim Curves in Terms of its Timelike Horizontal Biharmonic Partner Curves in the Lorentzian Heisenberg Group Heis^3

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Abstract: In this paper, we study Mannheim curves in the Lorentzian Heisenberg group Heis^3. We characterize Mannheim curves in terms of its timelike horizontal biharmonic partner curves in the Lorentzian Heisenberg group Heis^3.

Mathematics Subject Classification (2000): 58E20
Key words: Heisenberg group · Biharmonic curve · Mannheim curves

INTRODUCTION

The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: A space curve is a Mannheim curve if and only if its curvature and torsion satisfy the relation.

$$\kappa(s) = \lambda(\kappa^2(s)+\tau^2(s))$$

for some constant λ.

Harmonic maps $f:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy [1-7].

$$E(f) = \frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \text{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [8], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy [9].

Jiang derived the first and the second variation formula for the bienergy in [10], showing that the Euler--Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathrm{J}^f(\tau(f)) = -\Delta\tau(f) - \text{trace}R^N(df,\tau(f))df$$
$$= 0 \tag{1.4}$$

Where J^f is the Jacobi operator of f. The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps [11-18].

In this paper, we study Mannheim curves in the Lorentzian Heisenberg group Heis^3. We characterize Mannheim curves in terms of its timelike horizontal biharmonic partner curves in the Lorentzian Heisenberg group Heis^3.

The Lorentzian Heisenberg Group Heis^3: The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z) = (\overline{x}+x, \overline{y}+y, \overline{z}+z-\overline{x}y+x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.
The Lorentz metric g is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2$$

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The Lie algebra of Heis has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2,\mathbf{e}_3] = 2\mathbf{e}_1, [\mathbf{e}_3,\mathbf{e}_1] = 0, [\mathbf{e}_2,\mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1) = g(\mathbf{e}_2,\mathbf{e}_2) = 1, \; g(\mathbf{e}_3,\mathbf{e}_3) = -1.$$

Proposition 2.1: *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = -g(R(X, Y)Z, W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a,\mathbf{e}_b)\mathbf{e}_c, R_{abcd} = R(\mathbf{e}_a,\mathbf{e}_b,\mathbf{e}_c,\mathbf{e}_d),$$

Where the indices *a, b, c* and *d* take the values 1,2 and 3.

$$\begin{aligned} R_{232} &= 3R_{131} = -e_3, \\ R_{133} &= R_{122} = -e_1, \\ R_{233} &= R_{121} = -3e_2, \end{aligned}$$

and

$$R_{1212} = -1, R_{1313} = -1, R_{2323} = -3, \tag{2.3}$$

Timelike Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3: Let $\gamma: I \to Heis^3$ be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let {**T, N, B**} be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

T is the unit vector field γ' tangent to γ, **N** is the unit vector field in the direction of $\nabla_T T$ (normal to γ) and **B** is chosen so that {**T, N, B**} is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

Where κ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned}$$

Theorem 3.1: *(see [18]) Let $\gamma: I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is a timelike non-geodesic biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \kappa^2 - \tau^2 &= -1 + 4B_1^2, \\ \tau' &= -2N_1B_1. \end{aligned} \tag{3.2}$$

Corollary 3.2. *(see [18]) Let $\gamma: I \to Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \tau &= \text{constant}, \\ N_1 B_1 &= 0, \\ \kappa^2 - \tau^2 &= -1 + 4B_1^2. \end{aligned} \tag{3.3}$$

Theorem 3.3: *(see [18]) Let $\gamma: I \to Heis^3$ be a non-geodesic timelike curve on Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 \neq 0$ then γ is not biharmonic.*

Theorem 3.4: *(see [18]) Let $\gamma: I \to Heis^3$ be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 \neq 0$, then*

$$\mathbf{T}(s) = \sinh\phi_0\mathbf{e}_1 + \cosh\phi_0 \sinh\psi(s)\mathbf{e}_2 + \cosh\phi_0 \cosh\psi(s)\mathbf{e}_3, \tag{3.4}$$

Where $\phi_0 \in \mathbb{R}$.

Mannheim Curves In The Lorentzian Heisenberg Group Heis^3: Consider a nonintegrable 2-dimensional distribution $(x, y) \to H_{(x, y)}$ in $Heis^3$ defined as $H = \ker\omega$, where ω is a 1-form on $Heis^3$. The distribution H is called the horizontal distribution.

A curve $s \to \gamma(s) = (x(s), y(s), z(s))$ is called horizontal curve if $\gamma'(s) \in H_{\gamma(s)}$, for every s.

Lemma 4.1: *Let $\gamma : I \to Heis^3$ be a horizontal curve and ω is a 1-form on $Heis^3$ Then,*

$$\omega(\gamma'(s)) = 0. \tag{4.1}$$

Proof: We use the equation of γ,

$$\gamma'(s) = x'(s)\partial_x + y'(s)\partial_y + z'(s)\partial_z \tag{4.2}$$

From (2.1) we have

$$\frac{\partial}{\partial x} = e_3, \frac{\partial}{\partial y} = e_2 + xe_3, \frac{\partial}{\partial z} = e_1. \tag{4.3}$$

Substituting (4.3) into (4.2) we obtain

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s))\partial_z$$

Since γ is assumed to be a non-geodesic horizontal curve we have (4.1).

Lemma 4.2: *$\gamma : I \to Heis^3$ be a horizontal curve if and only if*

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 + \omega(\gamma'(s)) = z'(s) + x(s)\, y'(s) \tag{4.4}$$

If $\gamma(s)$ is horizontal curve, then we have

$$\gamma'(s) = x'(s)e_3 + y'(s)e_2 = x'(s)\frac{\partial}{\partial x} + y'(s)\frac{\partial}{\partial y} - x(s)y'(s)\frac{\partial}{\partial z}. \tag{4.5}$$

Using (2.1) and (4.5) we obtain

$$T = T_3\frac{\partial}{\partial x} + T_2\frac{\partial}{\partial y} + (T_1 - x(s)T_2)\frac{\partial}{\partial z}. \tag{4.6}$$

Definition 4.3: *Let $\gamma : I \to Heis^3$ be a unit speed non-geodesic curve. If there exists a corresponding relationship between the space curves γ and β such that, at the corresponding points of the curves, the principal normal lines of β coincides with the binormal lines of β, then β is called a Mannheim curve and γ a Mannheim partner curve of β. The pair $\{\gamma, \beta\}$ is said to be a Mannheim pair.*

Theorem 4.4: *Let $\beta : I \to Heis^3$ be a Mannheim curve and γ its timelike horizontal biharmonic partner curve. Then, the parametric equation of Mannheim curve β in terms of its timelike horizontal biharmonic partner curve γ of β are*

$$x_\beta(s) = \lambda\sinh[\kappa s+\rho]\cosh[\kappa s+\rho](\sinh[\kappa s+\rho]+\rho_1 s+\rho_2) -$$
$$\lambda(\cosh[\kappa s+\rho] - \left(\frac{1}{\kappa}\sinh[\kappa s+\rho]+\rho_0\right)\sinh[\kappa s+\rho])\cosh[\kappa s+\rho] + \frac{1}{\kappa}\sinh[\kappa s+\rho]+\rho_0,$$

$$y_\beta(s) = \lambda\sinh[\Re s+\rho](\cosh[\Re s+\rho] - \left(\frac{1}{\Re}\sinh[\Re s+\rho]+\rho_0\right)\sinh[\Re s+\rho]) - \lambda\cosh^2[\Re s+\rho].(\sinh[\Re s+\rho]+\rho_1 s+\rho_2)+$$
$$\frac{1}{\Re}\cosh[\Re s+\rho]+\rho_4,$$

$$z_\beta(s) = \lambda\cosh[\Re s+\rho]\cosh[\Re s+\rho] - \lambda\sinh[\Re s+\rho]\sinh[\Re s+\rho]$$
$$+\frac{1}{\Re}\sinh[\Re s+\rho] - \frac{1}{\Re}(-\frac{s}{2}+\frac{\sinh 2[\Re s+\rho]}{4\Re}) - \frac{\rho_0}{\Re}\cosh[\Re s+\rho]+\rho_5, \tag{4.7}$$

Where ρ, ρ_0, ρ_1, ρ_2, ρ_4, ρ_5, are constants of integration.

Proof: The covariant derivative of the vector field T is:

$$\nabla_{\mathbf{T}}\mathbf{T} = T_1'\mathbf{e}_1 + (T_2' + 2T_1T_3)\mathbf{e}_2 + (T_3' + 2T_1T_2)\mathbf{e}_3. \tag{4.8}$$

From (3.4), we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (\psi'\cosh\phi_0\cosh\psi(s) + 2\sinh\phi_0\cosh\phi_0\cosh\psi(s))\mathbf{e}_2 + (\psi'\cosh\phi\sinh\psi(s) + 2\sinh\phi_0\cosh\phi_0\sinh\psi(s))\mathbf{e}_3. \tag{4.9}$$

Since $|\nabla_T T|$ we obtain

$$\psi(s) = (\pm\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)s + \rho, \tag{4.10}$$

Where $\rho \in$ R.
Thus (3.4) and (4.10), imply

$$\mathbf{T} = \sinh\phi_0 e_1 + \cosh\phi_0\sinh[\Re s + \rho]e_2 + \cosh\phi_0\cosh[\Re s + \rho]e_3, \tag{4.11}$$

Where $\Re = (\frac{\kappa}{\cosh\phi_0} - 2\sinh\phi_0)$.

On the other hand, using (4.5) and (4.6) we have

$$T_1 = \sinh\phi_0 = 0. \tag{4.12}$$

Thus, we choose

$$\cosh\phi_0 = 1,\ \Re = \kappa. \tag{4.13}$$

Using (3.1) in (4.11), we obtain

$$\mathbf{T} = (\cosh[\kappa s + \rho], \sinh[\kappa s + \rho], \cosh[\kappa s + \rho]) - x(s)\sinh[\Re s + \rho]).$$

From (2.1), we get

$$\mathbf{T} = (\cosh[\kappa s + \rho], \sinh[\kappa s + \rho], \cosh[\kappa s + \rho]) - \left(\frac{1}{\kappa}\sinh[\kappa s + \rho] + \rho_0\right)\sinh[\kappa s + \rho]),$$

Where ρ_0 is constant of integration.
On the other hand, suppose that $\beta(s)$ is a Mannheim curve. Then by the definition we can assume that

$$\beta(s) = \gamma(s) + \lambda B(s) \tag{4.14}$$

From (3.1) and (4.11), we get

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\cosh[\kappa s + \rho]\mathbf{e}_2 + \kappa\sinh[\kappa s + \rho]\mathbf{e}_3.$$

Where $\Re\ \kappa$
By the use of Frenet formulas, we get

$$\begin{aligned}\mathbf{N} &= \frac{1}{\kappa}\nabla_{\mathbf{T}}\mathbf{T}\\ &= \cosh[\kappa s + \rho]\mathbf{e}_2 + \sinh[\kappa s + \rho]\mathbf{e}_3.\end{aligned} \tag{4.15}$$

Substituting (2.1) in (4.15), we have

$$\mathbf{N} = (\sinh[\kappa s + \rho], \cosh[\kappa s + \rho], \cosh[\kappa s + \rho](\sinh[\kappa s + \rho] + \rho_1 s + \rho_2)).$$

Noting that T×N = B, we have

$$\begin{aligned}\mathbf{B} = (&\sinh[\kappa s + \rho]\cosh[\kappa s + \rho](\sinh[\kappa s + \rho] + \rho_1 s + \rho_2) - (\cosh[\kappa s + \rho] - \left(\frac{1}{\kappa}\sinh[\kappa s + \rho] + \rho_0\right)\sinh[\kappa s + \rho])\cosh[\kappa s + \rho],\\ &\sinh[\kappa s + \rho](\cosh[\kappa s + \rho] - \left(\frac{1}{\Re}\sinh[\kappa s + \rho] + \rho_0\right)\sinh[\kappa s + \rho]) - \cosh[\kappa s + \rho]\cosh[\kappa s + \rho](\sinh[\kappa s + \rho] + \rho_1 s + \rho_2),\\ &\cosh[\kappa s + \rho]\cosh[\kappa s + \rho] - \sinh[\kappa s + \rho]\sinh[\Re s + \rho]).\end{aligned} \tag{4.16}$$

Next, we substitute (4.11) and (4.16) into (4.14), we get (4.7). The proof is completed.

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