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On the Behavaİor of Solutİons of the System of Ratİonal Dİfference Equatİons

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}-1}$$

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Abstract: In this paper, we investigate the solutions of the system of difference equations

$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}-1}$ Where $y_0, y_{-1}, x_0, x_{-1} \in \mathbb{R}$

Key words: Difference equation · Difference equation systems · Solutions

INTRODUCTION

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. There are many papers with related to the difference equations system for example,

In [1] Cinar studied the solutions of the systems of the difference equations.

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}$$

In [2] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations $x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}}$, $n = 0,1,2,\ldots,p,q$.

In [3] Papaschinnopoulos and Schinas proved the boundedness, persistence, the oscillatory behavior and the asymptotic behavior of the positive solutions of the system of difference equations.

$$x_{n+1}=\sum_{i=0}^{k}\frac{A_i}{y_{n-i}^{p_i}},\quad y_{n+1}=\sum_{i=0}^{k}\frac{B_i}{x_{n-i}^{q_i}}$$

In [4, 5] Özban studied the positive solutions of the system of rational difference equations $x_n=\frac{a}{y_{n-3}},\quad y_{n+1}=\frac{by_{n-3}}{x_{n-q}y_{n-q}}$ and

$$x_{n+1}=\frac{a}{y_{n-k}},\quad y_{n+1}=\frac{y_n}{x_{n-m}y_{n-m-k}}.$$

In [6, 7] Clark and Kulenović investigate the global asymptotic stability $x_{n+1}=\frac{x_n}{a+cy_n},\quad y_{n+1}=\frac{y_n}{b+dx_n}$.

In [8] Camouzis and Papaschinnopoulos studied the global asymptotic behavior of positive solutions of the system of rational difference equations.

$$x_{n+1}=1+\frac{x_n}{y_{n-m}},\quad y_{n+1}=1+\frac{y_n}{x_{n-m}}$$

In [9] Yang, Liu and Bai consided the behavior of the positive solutions of the system of the difference equations

$$x_n=\frac{a}{y_{n-p}},\quad y_n=\frac{by_{n-p}}{x_{n-q}y_{n-q}}$$

In [10] Kulenović, Nurkanović studied the global asymptotic behavior of solutions of the system of difference equations.

$$x_{n+1}=\frac{a+x_n}{b+y_n},\quad y_{n+1}=\frac{c+y_n}{d+z_n},\quad z_{n+1}=\frac{e+z_n}{f+x_n}$$

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In [11] Zhang, Yang, Megson and Evans investigated the behavior of the positive solutions of the system of difference equations.

$$x_n = A + \frac{1}{y_{n-p}}, \quad y_n = A + \frac{y_{n-1}}{x_{n-r}y_{n-s}}$$

In [12] Zhang, Yang, Evans and Zhu studied the boundedness, the persistence and global asymptotic stability of the positive solutions of the system of difference equations.

$$x_{n+1} = A + \frac{y_{n-m}}{x_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{y_n}$$

In [13] Yalcinkaya and Cinar studied the global asmptotic stability of the system of difference equations.

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}$$

In [14] Yalcinkaya, Cinar and Atalay investigated the solutions of the system of difference equations.

$$x_{n+1}^{(1)} = \frac{x_n^{(2)}}{x_n^{(2)} - 1}, \quad x_{n+1}^{(2)} = \frac{x_n^{(3)}}{x_n^{(3)} - 1}, \ldots, x_{n+1}^{(k)} = \frac{x_n^{(1)}}{x_n^{(1)} - 1}$$

In [15] Yalcinkaya studied the global asmptotic stability of the system of difference equations.

$$z_{n+1} = \frac{t_n z_{n-1} + a}{t_n + z_{n-1}}, \quad t_{n+1} = \frac{z_n t_{n-1} + a}{z_n + t_{n-1}}$$

In [16] Irićanin and Stević studied the positive solutions of the system of difference equations.

$$x_{n+1}^{(1)} = \frac{1 + x_n^{(2)}}{x_{n-1}^{(3)}}, \qquad x_{n+1}^{(2)} = \frac{1 + x_n^{(3)}}{x_{n-1}^{(4)}}, \ldots, \qquad x_{n+1}^{(k)} = \frac{1 + x_n^{(1)}}{x_{n-1}^{(2)}}$$

$$x_{n+1}^{(1)} = \frac{1 + x_n^{(2)} + x_{n-1}^{(3)}}{x_{n-2}^{(4)}}, \quad x_{n+1}^{(2)} = \frac{1 + x_n^{(3)} + x_{n-1}^{(4)}}{x_{n-2}^{(5)}}, \ldots, x_{n+1}^{(k)} = \frac{1 + x_n^{(1)} + x_{n-1}^{(2)}}{x_{n-2}^{(3)}}$$

Also see [15, 17-20].

In this paper, we investigate the behavaior of the solutions of the difference equation system.

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1} \tag{1.1}$$

Where the initial conditions are arbitrary real numbers. This paper was motivated by [21] and [22].

Teorem 1: Let $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$ be arbitrary real numbers and let $\{x_n, y_n\}$ be a solutions of the system (1.1). Also, assume that $ad \neq$ and $cd \neq 1$ Than all solutions of (1.1) are

$$x_n = \begin{cases} \dfrac{d}{(ad-1)^n}, & n - odd \\ c(cb-1)^n, & n - even \end{cases} \tag{1.2}$$

$$y_n = \begin{cases} \dfrac{b}{(cb-1)^n}, & n - odd \\ a(ad-1)^n, & n - even \end{cases} \tag{1.3}$$

Proof: For $n = 0,1,2,3$ we have

$$x_1 = \frac{x_{-1}}{y_0 x_{-1} - 1} = \frac{d}{ad - 1}$$

$$y_1 = \frac{y_{-1}}{x_0 y_{-1} - 1} = \frac{b}{cb - 1}$$

$$x_2 = \frac{x_0}{y_1 x_0 - 1} = \frac{c}{\frac{b}{cb-1}c - 1} = c(cb-1)$$

$$y_2 = \frac{y_0}{x_1 y_0 - 1} = \frac{a}{\frac{d}{ad-1}a - 1} = a(ad-1)$$

$$x_3 = \frac{x_1}{y_2 x_1 - 1} = \frac{\frac{d}{ad-1}}{a(ad-1)\frac{d}{ad-1} - 1} = \frac{d}{(ad-1)^2}$$

$$y_3 = \frac{y_1}{x_2 y_1 - 1} = \frac{\frac{b}{cb-1}}{c(cb-1)\frac{b}{cb-1} - 1} = \frac{b}{(cb-1)^2}$$

for $n = k$ assume that

$$x_{2k-1} = \frac{d}{(ad-1)^k}$$

$$x_{2k} = c(cb-1)^k$$

$$y_{2k-1} = \frac{b}{(cb-1)^k}$$

and

$y_{2k} = a(ad-1)^k$

are true. Then for $n = k+1$ we will show that (1.2) and (1.3) are true. From (1.1), whe have

$$x_{2k+1} = \frac{x_{2k-1}}{y_{2k} x_{2k-1} - 1} = \frac{\frac{d}{(ad-1)^k}}{a(ad-1)^k \frac{d}{(ad-1)^k} - 1} = \frac{\frac{d}{(ad-1)^k}}{\frac{(ad-1)}{1}} = \frac{d}{(ad-1)^{k+1}}$$

Also, smilarly from (1.1), we have

$$y_{2k+1} = \frac{y_{2k-1}}{x_{2k} y_{2k-1} - 1} = \frac{\frac{b}{(cb-1)^k}}{c(cb-1)^k \frac{b}{(cb-1)^k} - 1} = \frac{b}{(cb-1)^{k+1}}$$

Also, we have

$$x_{2k+2} = \frac{x_{2k}}{y_{2k+1} x_{2k} - 1} = \frac{c(cb-1)^k}{\frac{b}{(cb-1)^{k+1}} c(cb-1)^k - 1} = \frac{c(cb-1)^k}{\frac{cb-cb+1}{cb-1}} = c(cb-1)^{k+1}$$

$$y_{2k+2} = \frac{y_{2k}}{x_{2k+1} y_{2k} - 1} = \frac{a(ad-1)^k}{\frac{d}{(ad-1)^{k+1}} a(ad-1)^k - 1} = \frac{a(ad-1)^k}{\frac{(ad-ad+1)}{ad-1}} = a(ad-1)^{k+1}$$

□

Teorem 2. Let $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$ be arbitrary real numbers and let $\{x_n, y_n\}$ be a solutions of the system (1.1). If $0 < a,b,c,d < 1$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \pm\infty$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

Proof: From $0 < a,b,c,d < 1$ we have $0 < ad < 1 \Rightarrow - 10$ and $0 < cb < 1 \Rightarrow -1 < cb - 1 < 0$. Hence, we obtain

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.\infty = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

and

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.\infty = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

Smilarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c \lim_{n\to\infty} (cb-1)^n = c.0 = 0$$

and

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

$\square$

Teorem 3. Let $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$ be arbitrary real numbers and let $\{x_n, y_n\}$ be a solutions of the system (1.1). If $1 < ad,cb < 2$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \pm\infty$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

Proof: From $1 < ad,cb < 2$ we have $1 < ad < 2 \Rightarrow 0 < ad-1 < 1$ and $1 < cb < 2 \Rightarrow 0 < cb - 1 < 1$. Hence, we obtain

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.\infty = \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}$$

and

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.\infty = \begin{cases} -\infty, & b < 0 \\ +\infty, & b > 0 \end{cases}$$

Smilarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c \lim_{n\to\infty} (cb-1)^n = c.0 = 0$$

and

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

$\square$

Teorem 4: Let $\{x_n, y_n\}$ be a solutions of (1.1). If $ad,cb \in (-\infty, -1)$ and $ad,cb \in (2, +\infty)$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = 0$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = \infty$$

Proof: From $-\infty < ad < 0 \Rightarrow -\infty < ad - 1 < -1$ we have

$$\lim_{n\to\infty} (ad-1)^n = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

and from $-\infty < cb < 0 \Rightarrow -\infty < cb - 1 < -1$ we have

$$\lim_{n\to\infty} (cb-1)^n = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.0 = 0$$

and

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.0 = 0$$

Smilarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c \lim_{n\to\infty} (cb-1)^n = c.\infty = \begin{cases} -\infty, & c>0 \text{ and } n-odd \\ +\infty, & c<0 \text{ and } n-odd \\ +\infty, & c>0 \text{ and } n-even \\ -\infty, & c<0 \text{ and } n-even \end{cases}$$

and

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.\infty = \begin{cases} -\infty, & a>0 \text{ and } n-odd \\ +\infty, & a<0 \text{ and } n-odd \\ +\infty, & a>0 \text{ and } n-even \\ -\infty, & a<0 \text{ and } n-even \end{cases}$$

□

Teorem 5: Let $\{x_n, y_n\}$ be a solutions of (1.1). If $a,b,c,d \in \mathbb{R}$ and $-1 < a,b,c,d < 0$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \infty$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

Proof: From $-1 < a,b,c,d < 0$ we obtain $0 < ad < 1 \Rightarrow -1 < ad-1 < 0$ and $0 < cb < 1 \Rightarrow -1 < cb-1 < 0$ and we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.\infty = \begin{cases} +\infty, & n-odd \\ -\infty & n-even \end{cases}$$

and

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.\infty = \begin{cases} +\infty, & n-odd \\ -\infty & n-even \end{cases}$$

smilarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cb-1)^n = c \lim_{n\to\infty} (cb-1)^n = c.0 = 0$$

and

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

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